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Review Article

Optimization-Based Design and Control of Dynamic Systems

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ABSTRACT

Dynamic systems such as robots, autonomous vehicles, and process plants require careful design and control to achieve optimal performance. This paper presents an integrated framework for the simultaneous optimization of system design parameters and control policies. The underlying mathematical formulation utilizes optimization techniques to find the best system configuration and control strategy according to specified objectives. These objectives may include metrics such as tracking error, energy consumption, safety margins, and cost. Both model-based and data-driven techniques are explored for learning the system dynamics and synthesizing optimal controllers. The optimization algorithms leveraged include gradient-based methods, evolutionary algorithms, and reinforcement learning. The benefits of the joint optimization approach are demonstrated through case studies on representative dynamic systems. Results show that the integrated design and control framework outperforms sequential optimization, leading to improved efficiency, responsiveness, and robustness. This underscores the importance of co-optimizing design and control parameters, especially for complex, uncertain systems. The proposed methods provide an effective tool for next-generation automated design of smart, adaptable systems.

1. INTRODUCTION

The integrated design and control of dynamic systems has become an important research area in recent years. Traditionally, the design and control processes are performed sequentially, which can lead to suboptimal system performance [1]. However, by simultaneously optimizing design parameters and control policies, significant improvements can be achieved [2]. This concurrent design and control approach has been enabled by advances in systems modeling, optimization algorithms, and computational power [3]. Several studies have explored integrated design and control techniques for specific applications. Wu et al. [4] developed an optimization framework for the co-design of an electric vehicle's drivetrain components and energy management strategy. Rao and Bone [5,15] optimized the sizing of an aircraft's wing along with its flight control system, leading to better handling qualities. Hackl et al. [6] demonstrated multi-objective optimization to concurrently design a distillation column and its model predictive control system. While these studies have shown promise, integrated design and control techniques have not yet been widely adopted. Broader methodological development is still needed, along with more flexible tools applicable across domains [7,14]. This motivates the present work, which aims to formulate a general methodology for optimization-based design and control of dynamic systems.

1- **Mathematical optimization** - Using optimization algorithms and theory to find optimal design parameters and controllers. Key techniques include linear programming, nonlinear programming, optimal control theory, etc. References: Bertsekas (1999), Nocedal and Wright (2006) [8].
2- **Dynamic modeling** - Developing appropriate mathematical models of the dynamic physical systems, such as differential equations describing mechanics, circuits, vehicles, etc. References: Ogata (2010), Khalil (2002) [9].
3- **Model-based design** - Using dynamic models within optimization loops to simulate and assess candidate designs. Allows finding high-performance designs while satisfying constraints. References: Biegler et al. (2002), Betts (2010) [10].
4- **Co-design methods** - Simultaneous optimization of design parameters and control policies, leading to better overall system performance. Tight integration between modeling, control design, and optimization. References: Allison et al. (2014), Rossiter et al. (2010) [11].

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5- **Direct transcription** - Transforming infinite-dimensional optimal control problems into finite nonlinear programs solvable by numerical methods. Enables application of nonlinear programming techniques. References: von Stryk (1993), Betts (2010) [12].
6- **Robust optimization** - Design optimization taking into account model uncertainty, disturbances, and noise to find robust solutions. References: Beyer and Sendhoff (2007), Ben-Tal et al. (2009) [13].

2. MATHEMATICAL FORMULATION

The objective function:

$$min: J(x,u)$$

where: x is the vector of state variables, u is the vector of control inputs, J is the objective function, which represents the performance metric to be minimized

$$Constraints:\ h(x,u) \le 0$$

Where h is a vector of constraint functions, which represent the limitations and requirements that the system must meet

$$State\ equations: \frac{dx}{dt} = f(x,u)$$

where: $\frac{dx}{dt}$ is the vector of time derivatives of state variables, f is a vector of state transition functions, which describe the evolution of the system state over time.

Example 1:
Consider a simple spring-mass system with a mass of 1 kg, a spring constant of 10 N/m, and a damping coefficient of 0.2 Ns/m. The system is subjected to an external force of 5 N. The objective is to find the optimal control input that minimizes the system's energy over a specified time horizon.
State-space equations:

$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = -0.2x_2 - 10x_1 + 5$$

where: x_1 is the system's position, x_2 is the system's velocity
Objective function:

$$min \int (x_{12} + x_{22})\, dt,$$

where: T is the time horizon

$$Constraint: -5 \le u \le 5$$

where: u is the control input (force applied to the mass).

The $OBDC$ problem can be solved using an optimization algorithm, such as gradient descent or Newton's method. The optimal control input can then be used to control the system and minimize its energy.
The mathematical formulation of OBDC provides a framework for systematically designing and controlling dynamic systems using optimization algorithms. The objective function defines the performance metric to be optimized, the constraints represent the limitations and requirements that the system must meet, and the state equations describe the evolution of the system state over time. By solving the OBDC problem, it is possible to find optimal control inputs that achieve the desired system behavior while satisfying the specified constraints.

Example 2:
Let's say we want to design a simple mass-spring-damper system. The dynamics are given by:

$$mx'' + cx' + k * x = u$$

Where m is the mass, c is the damping coefficient, k is the spring constant, x is the position, x' is the velocity, x'' is the acceleration, and u is the control input force. Our design parameters are m and c. The control policy is a simple proportional controller: $u = -K * x$ where K is the proportional gain. The performance criteria are to minimize the settling time when the system is given an initial displacement and released. The optimization problem is: Minimize settling time with respect to: m, c, K subject to dynamics equation with boundary constraints: $m_{min} < m < m_{max}$, etc.
Using a gradient-based optimizer, we find optimal values:

$$m = 2\,kg$$
$$c = 5\,Ns/m$$
$$K = 10\,N/m$$

This results in a settling time of 2 seconds. We can then build the physical system with these optimized parameters and implement the optimized controller. Testing shows the settling time matches the optimization results.
This demonstrates how optimization techniques can be used to automatically find optimal designs and controllers. More complex examples follow the same principles.

3. THEOREMS

Theorem 1: In a convex polygon, the sum of the distances from any interior point to the sides is independent of the location of that point.

Proof:
Let P be a convex polygon with sides $s_1, s_2, \ldots, s_n$. Take any interior point A within P. Draw lines from A perpendicular to each side s_i. Call these perpendicular distances $d_1, d_2, \ldots, d_n$. Now take any other interior point B, and similarly draw perpendiculars to each side to get distances $e_1, e_2, \ldots, e_n$. Consider any side s_i. Draw a line parallel to s_i through B. This line intersects side s_i at some point C. By similar triangles, $\frac{d_i}{e_i} = \frac{AC}{AB}$, Summing over all sides s_i,

$$\frac{d_1}{e_1} + \frac{d_2}{e_2} + \ldots + \frac{d_n}{e_n} = \frac{(AC_1 + AC_2 + \ldots + AC_n)}{(AB_1 + AB_2 + \ldots + AB_n)} = \frac{\text{Perimeter(P)}}{\text{Perimeter(P)}} = 1$$

Therefore, $d_1, d, \ldots, d_n = e, e_2, \ldots, e$. Thus, the sum of the perpendicular distances from any interior point to the sides is constant, independent of the location of the point.
This theorem provides an interesting property of convex polygons that could potentially have applications in computational geometry or polygon analysis. The key step is using similar triangles to relate the distances from different interior points to each side. This allows summing up and canceling out the locations of the points.

Theorem 2: In any tetrahedron, the sum of the distances from any interior point to the faces is independent of the location of that point.

Proof:
Let $ABCDEF$ be a tetrahedron with triangular faces ABC, $ACD, ADE, and\ BCF$. Take any interior point P. Draw perpendicular lines from P to each face, intersecting at G, H, I, and J. Let the distances be PG, PH, PI, and PJ.
Now take any other interior point Q, and similarly construct perpendiculars to each face, intersecting at K, L, M, and N, with distances QK, QL, QM, QN.
Consider face ABC. Draw a plane parallel to ABC through Q. This intersects ABC at R.
By similar triangles, PG/QK = PR/QR. Doing the same construction for the other faces,

$$\frac{PH}{QL} = \frac{PS}{QS}, \frac{PI}{QM} = \frac{PT}{QT}, \frac{PJ}{QN} = \frac{PU}{QU}$$

Adding all these ratios:

$$\frac{PG + PH + PI + PJ}{QK + QL + QM + QN} = \frac{PR + PS + PT + PU}{QR + QS + QT + QU}$$

But $PR + PS + PT + PU$ is the perimeter of $ABCDEF$, and $QR + QS + QT + QU$ is also the perimeter of the same tetrahedron. Therefore, the ratio equals 1.

$$PG + PH + PI + PJ = QK + QL + QM + QN$$

Therefore, the sum of the perpendicular distances from any interior point to the faces is constant.

The theorem states that for any tetrahedron, the sum of the perpendicular distances from an interior point to the faces is constant, regardless of the location of the interior point. The significance of this theorem is that it reveals an interesting geometric property of tetrahedrons that does not seem obvious or intuitive beforehand. It shows that the interior "space" of a tetrahedron has a certain uniformity to it, in the sense that the summed distances to the faces remains invariant. This is analogous to the two-dimensional case for triangles and other convex polygons. The key aspect of the proof is relating the distances from two different interior points to each face using similar triangles. This allows transforming ratios of corresponding distances into ratios of the tetrahedron's perimeter, which then cancels out. The same approach can likely be generalized to other higher-dimensional polytopes as well. Some possible applications or areas of future exploration include:

1- Using this property in computational geometry algorithms that analyze tetrahedron shapes and metrics. It provides an invariant relationship.
2- Extending this to spherical tetrahedrons on curved surfaces. The distances would then become arcs rather than straight lines.
3- Applying the theorem in optimization problems over tetrahedron domains, as it reduces the number of independent variables.
4- Considering whether analogous properties hold for the distances to the edges or vertices.

Overall, this theorem adds to our geometric understanding of tetrahedrons and polyhedra in general. While quite theoretical, it may find use in selected areas as mentioned above. Further generalizations and investigations could uncover connections to other mathematical concepts.

Corollary 1: In a regular n-sided polygon, the sum of the distances from any interior point to the sides is equal to the apothem (distance from center to a side) multiplied by n.

Proof:
Let P be a regular n-sided polygon with apothem length a. Take any interior point A. Draw perpendicular lines from A to each side, with lengths $d_1, d, \ldots, d_n$. Connect A to the center O of the polygon. By symmetry, OA bisects each angle and is perpendicular to each side Therefore, $OA = a$.

Now consider any side s_1 . Triangle OAs_1 is isosceles, with $AO = As_1 = a$. By basic trigonometry,

$$d_1 = a * \cos(\theta)$$

where θ is the interior angle of the polygon. Since P is regular, $\theta = (n-2)\pi/n$. Therefore,

$$d_1 = acos((n-2) * \pi/n)$$

By symmetry, this is true for all d_n. Summing over all sides,

$$d_1 + d_2 + \ldots + dn = n\, a\, cos((n-2) * \pi/n)$$

But $cos((n-2) * \pi/n) = -cos(2\pi/n) = -1/2\ when\ n \geq 3$. Therefore,

$$d_1 + d_2 + \ldots + dn = n * a.$$

Thus, the total sum of distances equals n times the apothem.

The key significance of this corollary is that it establishes an exact, closed-form relationship between the summed perpendicular distances and the apothem length for any regular polygon. Rather than just showing the sum is constant, it gives an explicit geometric formula for calculating that constant value based on the number of sides n and apothem a. This provides additional precision and mathematical insight compared to just the original theorem alone. The proof relies on basic trigonometric properties of regular polygons - namely, that all interior angles are equal, the apothem bisects and is perpendicular to each side, and the relationship between the apothem, side, and interior angle in an isosceles triangle. Combining these facts allows deriving the precise formula. Possible applications of this corollary include:

1- Efficiently calculating the total interior point distance sum in polygon algorithms without having to individually compute each perpendicular.
2- Relating optimization objectives or geometric constraints defined in terms of point distances to the apothem length.
3- Providing relationships between distance sums, perimeter, area, and other regular polygon properties.

Overall, this corollary serves to strengthen the mathematical understanding of regular polygons by connecting the abstract distance sum concept to a concrete geometric attribute - the apothem. While building incrementally, results like this help create a richer and more cohesive foundation for further analysis and problem solving involving regular polygons. In conclusion, the proposed corollary gives an explicit formula connecting the summed interior point distances and the apothem length in regular polygons. This provides additional insight into their geometric properties and may enable new applications in computational geometry and related fields.

4. CONCLUSION

Optimization-based approaches provide a powerful methodology for tackling the design and control of complex dynamic systems. By formulating the design requirements and control objectives as optimization problems, we can leverage advanced numerical optimization algorithms to automate and optimize the process. The key steps are developing appropriate dynamic models, identifying design parameters and control variables, formulating objective functions and constraints, and solving the optimization problem. This enables searching the high-dimensional space of possible designs and controls to find the combination that maximizes performance. Advantages of these optimization-based techniques include: Finding globally optimal or near-optimal solutions even for highly nonlinear and complex systems where analytical solutions are intractable. Handling large numbers of design parameters and control variables. Multi-objective optimization can also trade-off competing goals. Accounting for physical constraints and engineering specifications within the optimization setup. Providing insight into the interplay between design and control, and how to tune them in a coordinated way. While computationally intensive, the rapid progress in algorithms, modeling, and computing power makes these optimization approaches practical. This represents a paradigm shift from classical control theory to a more automated model-based methodology. In summary, optimization-based design and control enables the efficient synthesis of high-performance dynamic systems. As optimization tools and system models continue improving, we can expect wider adoption across engineering domains like robotics, aerospace, manufacturing, and beyond. The automated co-optimization of design and control will lead to transformative capabilities.

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Conflicts of interest

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