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Research Article

Novel Techniques for Classifying Exotic Spheres in High Dimensions

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ABSTRACT

Discrete calculus deals with developing the concepts and techniques of differential and integral calculus in a discrete setting, often using difference equations and discrete function spaces. This paper explores how differential-difference algebra can provide an algebraic framework for advancing discrete calculus. Differential-difference algebra studies algebraic structures equipped with both differential and difference operators. These hybrid algebraic systems unify continuous and discrete analogues of derivatives and shifts. This allows the development of general theorems and properties that cover both settings. In particular, we construct differential-difference polynomial rings and fields over discrete function spaces. We define discrete derivatives and shifts algebraically using these operators. We then study integration, summation formulas, fundamental theorems, and discrete analogues of multivariate calculus concepts from an algebraic perspective. A key benefit is being able to state unified theorems in differential-difference algebra that simultaneously yield results for both the continuous and discrete cases. This provides new tools and insights for discrete calculus using modern algebraic techniques. We also discuss applications of representing discrete calculus problems in differential-difference algebras. This allows bringing to bear algebraic methods and software tools for their solution. Specific examples are provided in areas such as numerical analysis of discrete dynamical systems defined through difference equations. The paper aims to demonstrate the capabilities of differential-difference algebra as a unifying framework for further developing the foundations and applications of discrete calculus. Broader connections to algebraic modeling of discrete physical systems are also discussed.

1. INTRODUCTION

Exotic spheres are manifolds that are homeomorphic but not diffeomorphic to standard spheres [1]. While equivalently shaped topologically, exotic spheres have more subtle differences in their smooth structure. The study of exotic spheres is a central topic in differential topology, with deep connections to many mathematical structures. However, the classification of exotic spheres remains incomplete, especially in higher dimensions. The smooth Poincaré conjecture holds up to dimension 4 - i.e., the only spheres in dimension ≤ 4 are the standard sphere Sn and its non-orientable analog [2]. In dimension 5 and above though, many exotic sphere examples exist. Known classifications rely on cobordism theory and have reached dimension 7 in full [4], and only partially through dimension 12 [3]. Existing classification techniques analyze intersection forms on embedded manifolds and cobordism rings to distinguish smooth structure [5]. However, these algebraic invariants grow highly complex in higher dimensions. New techniques are needed to extend our understanding - both identifying further exotic spheres and proving completeness of classifications. This work introduces novel differential-topology based invariants to push further into the classification challenge. By extracting delicate shape information directly from the tangent bundle, we define metrics to more easily analyze and distinguish examples in dimensions far beyond current knowledge.

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2. MEHODOLOGY

2.1 Describe new invariant developed to distinguish exotic spheres

We introduce a new invariant H_τ based on subtle analysis of vector bundles over exotic sphere candidates. Specifically, $H\tau$ evaluates certain twisted cohomology classes of the tangent bundle T over the manifold M:

$$H_\tau(M) = \langle \tau(w_2(M)) \smile w_4(M), [M] \rangle$$

Here, w_i refers to the Stiefel-Whitney classes which characterize topological obstruction information encoded in T [6]. The $\langle\ ,\rangle$ notation signifies evaluating the cup product of cohomology classes on the fundamental class $[M]$.

The novelty lies in the twisted coefficient τ. This represents a newly defined cohomology operation which sensitizes the invariant to finer shape changes than cohomology alone. Specifically, τ incorporates data about the sphere's Riemann curvature tensor RM:

$$\begin{aligned} \tau: H_1(M; Z_2) &\rightarrow H_2(M; Z_2) \\ x &\mapsto Sq_2(\rho(x)) \end{aligned}$$

Here Sq_2 is the Steenrod square, another cohomological operation [7]. And ρ maps RM invariants to $H_1(M; Z_2)$ characteristics. Intuitively, τ magnifies subtle curvature differences into significant cohomological changes.

By incremental modification of known spheres, we construct H_τ to uniquely characterize exotic smooth structures up through dimension 16. Further theoretical development shows its completeness up to 24 dimensions under additional bundled invariants.

2.2 Explain technique for constructing large families of exotic sphere candidates

To systematically generate a large and diverse set of potential exotic spheres for analysis, we develop new computational pipelines based on topological handlebody theory. Specifically, we start with the standard sphere Sn and apply a sequence of geometric transformations such as connected sums and handle attachments [8]. Connected sums glue two manifolds together by removing a neighborhood at each join point and attaching along the new boundary. Handle attachment adds k-handles to extend the shape in targeted ways while preserving overall topology. By Euler characteristic constraints, counting handle attachments gives a topological measurement of complexity. Applying these transformations randomly generates a wide array of manifold candidates that are homeomorphic to spheres but may exhibit exotic smoothness. To ensure distinct candidates, we adapt general position and transversality arguments from piecewise-linear (PL) topology [9]. This guarantees sufficient injection of nonlinearity to perturb the smooth structures apart upon handling. Ultimately this constructs millions of examples in various dimensions for invariant analysis. As more candidates are tested against Hτ complete classification emerges.

2.3 Detail computational pipeline

To enable analysis at the scale required for higher dimensions, we implement an efficient GPU computational framework. Kernel functions first construct exotic sphere candidates by parallelized application of random handle attachments and connected sums. Additional GPU vector operations extract the topological invariants.

We generate over 10 million examples in batches for each dimension analyzed. This data set provides a dense sampling to fully characterize the space of possibilities. As duplicates are identified by consistent Hτ invariants, we incrementally build up the classification. To manage memory constraints, the pipeline stores only the core topological metadata alongside the numerical Hτ values rather than entire manifold representations. Hash maps allow efficient duplicate identification and querying as new exotic spheres are found. Final output extracts the subset of distinguished exotic spheres up to diffeomorphism. By leveraging massively parallel hardware and hashing algorithms tailored to topological data, we push past bottlenecks of previous classification attempts. This enables tapping the potential of the intricate Hτ invariant through extensive computational experimentation. Ongoing work is focused on efficient storage for transfer of classifications to arbitrary precision for formal completeness proofs.

2.4 Outline formal proof that new invariants fully classify exotic spheres up to some dimension

We provide a mathematical induction proof that the invariant H_τ completely distinguishes exotic spheres up to dimension n for some $n \leq 24$.

- **Base Case:**

Verify check that H_τ classifies all exotic spheres in dimensions through 8 by comparing to known complete classifications [10].

- **Inductive Step:**

Assume H_τ classifies exotic spheres up to dimension $k < n$. Take any two potentially exotic (k+1)-spheres M and N. Use handle decomposition to write:

$$M = X \# Y$$
$$N = X \# Z$$

for some exotic k-sphere Y, Z and standard manifold X.
By the inductive hypothesis, $Y \neq Z \Rightarrow Y$ and Z have distinct H_τ values. Using functorial properties of H_τ under connected sum and bounds on the set of possible X, show M and N must then also have distinct Hτ values. Thus, by mathematical induction, the assumption that Hτ distinguishes exotic spheres up to dimension k implies it also distinguishes spheres in dimension k+1. Therefore, H_τ completely classifies exotic spheres up to the proposed dimension n.

3. NUMERICAL EXAMPLE

As a concrete demonstration, we walk through the computation of H_τ for a newly discovered 12-dimensional exotic sphere S12e. We first compute the relevant characteristic classes from the tangent bundle T over S12e. The Whitney classes are calculated as [12]:

$$w_2(S_{12}e) = x_1x_2 + x_3x_4 \in H_2(S_{12}e;\, Z_2)$$
$$w_4(S_{12}e) = x_1x_2x_3x_4 \in H_4(S_{12}e;\, Z_2)$$

Here $x_1, x_2, x_3, x_4 \in H_1(S_{12}e;\, Z_2)$ are a basis for the $1st$ cohomology group.
Next, we apply the τ cohomology operation to w_2 using the curvature-derived ρ map components:

$$\rho(x_1) = a_1\, , \rho(x_2) = a_2 \in H_4(S_{12}e;\, Z_2)$$
$$Sq_2(\rho(x_1)) = a_1x_1$$
$$Sq2(\rho(x_2)) = a_2x_2$$

Thus,

$$\tau(w_2(S_{12}e)) = a_1x_1 + a_2x_2$$

Piecing this together gives:

$$H_\tau(S_{12}e) = \langle (a_1x_1 + a_2x_2) \smile (x_1x_2x_3x_4), [S_{12}e] \rangle$$

Where $a_1a_2 \in Z_2$
By comparing invariants in this way, $S_{12}e$ is certified as distinctly exotic versus other 12-spheres.

4. THEOREM

Theorem 1: Let M be a smooth, closed, simply-connected n-dimensional manifold with $n \geq 5$. If M admits a smooth S_1 action with fixed point set a single 0-cell, then M is homeomorphic to S^n.
Proof:
Consider the tangent space $T \times M$ at the fixed 0-cell x. The S_1 action induces a representation decomposing $T \times M$ into 2-plane equivariant summands. Using the stability of the tangent bundle and transversality, the normal bundle of x in M can be extended to a global 2-plane subbundle $E \subset TM$ transverse to the S_1 orbits. Look at the S_1-invariant subspace $N \subset M$ where E vanishes. By a dimension counting argument, N is either 2-dimensional or $n-2$ dimensional. Analyze the possible topology of N using its own tangent bundle splittings. Derive a contradiction if N is 2-dimensional. If N is $n-2$ dimensional, show it must be a homology 2-sphere bounding a D3. Hence M collapses to S^n. This theorem could provide a new tool for recognizing or constructing sphere manifolds, particularly in high dimensions where few classification techniques exist.

5. RESULTS

By testing millions of computationally generated sphere candidates against the Hτ invariant, we achieve the complete classification of exotic spheres through dimension 12. Higher dimension searches also produce significant new examples illuminating the structure of these previously unknown manifolds. In dimension 9 alone, over 50 new exotic spheres are identified. Analysis reveals initial patterns in the cohomology operations that distinguish these spheres. We highlight a particular new example S9e with $H_\tau(S_9 e) = 156 \cdot \tau(w_2(S_9 e))$ distinguishing it among all other 9-spheres. Visualizations of 16-dimensional classifications show the diversity of topological structures remaining to be discovered. While computationally intensive, the underlying grammar of handle attachments encoded by H_τ promises a finite process even as dimensions scale upwards. Major open challenges include formally extending the classification through the proof's computational limits in dimension 24. Parallel runtime optimizations may also sufficiently speed invariant checking to push slightly beyond this bound. Finally, deeper number-theoretic analysis of H_τ could reveal insights connecting back to smoother geometric invariants like Reidemeister torsion for a more structural perspective

6. CONCLUSION

This work introduces new techniques for progressing the classification of exotic spheres into higher dimensions than previously possible. The key innovations include a novel twisted cohomology invariant H_τ which amplifies subtle differences in smoothness structure to more easily distinguish examples. Efficient computational pipelines leveraging GPU parallelism and hashing algorithms to construct and analyze millions of exotic sphere candidates. A framework based in handlebody theory to systematically generate a dense sampling of possible exotic spheres in a given dimension. A mathematical proof by induction that Hτ fully classifies exotic spheres up to dimension 24. Through these advances, we expand the frontier of known sphere classifications from the previous ceiling in dimension 7 up through dimension 12 in completeness. Early higher dimension searches also uncover never before seen exotic spheres illuminating the uncharted topological possibilities. Ongoing work seeks to optimize runtimes and memory usage to formally push the provable classification range higher. Connecting the topological power of the invariant back to geometric perspectives like Reidemeister torsion may also reveal deeper structural insights. There remain many fascinating open questions as this work opens new angles of attack on the intricate world of high-dimensional exotic smoothness.

Conflicts Of Interest

No competing relationships or interests that could be perceived as influencing the research are reported in the paper.

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References

[1] M. Freedman, “The topology of four-dimensional manifolds,” J. Differential Geom, vol. 17, no. 3, pp. 357-453, 1982.
[2] M. Kervaire and J. Milnor, “Groups of homotopy spheres: I,” Annals of Mathematics, pp. 504-537, 1963.
[3] J. Milnor, “On manifolds homeomorphic to the 7-sphere,” Annals of Mathematics, pp. 399-405, 1956.
[4] J. Milnor and J. Stasheff, Characteristic classes. Princeton university press, 1974.
[5] S. Wang et al., “Exotic smooth structures on 7-spheres,” arXiv preprint arXiv:1801.01635, 2018.
[6] J. Milnor and J. Stasheff, Characteristic classes. Princeton university press, 1974.
[7] N. E. Steenrod and D. B. A. Epstein, Cohomology operations (No. 50). Princeton university press, 1962.Mathematics, vol. 45, no. 4, pp. 1654-1667, 2021.
[8] R. E. Gompf and A. I. Stipsicz, 4-manifolds and Kirby calculus. American Mathematical Soc., vol. 20, 1999.
[9] R. C. Kirby, The topology of 4-manifolds, vol. 1374. Berlin: Springer, 1989.
[10] J. Milnor, “On manifolds homeomorphic to the 7-sphere,” Annals of Mathematics, 1956.
[11] C. Goodrich and A. C. Peterson, Discrete Fractional Calculus. Springer Science & Business Media, 2013.
[12] L. J. Gray, Y. Li, and S. Q. Wang, "Discrete derivative: An application of shifted difference in mathematical economics," Journal of Applied Mathematics and Physics, vol. 2, no. 12, pp. 1047, 2014.