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Research Article

Delay Differential-Algebraic Equations (DDAEs)

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ABSTRACT

Delay differential-algebraic equations (DDAEs) are an important class of mathematical models that broaden standard differential-algebraic equations (DAEs) to incorporate discrete time delays. The time lag terms pose significant analytical and computational challenges. This paper provides a comprehensive overview of current and emerging methods for solving DDAEs and systems of DDAEs. Generalized Taylor series techniques, linear multistep methods, and reduction to ordinary differential equations are examined for numerically integrating DDAEs. Stability, convergence, and accuracy considerations are discussed to assess solver performance. Software libraries and custom implementation tools are also surveyed. Both theoretical analysis and practical application of algorithms are covered. Through definitions, examples, error analyses, and code demonstrations, this paper equips readers to understand key facets of DDAEs and employ advanced techniques to solve them. The topics presented here represent important progress toward addressing real-world systems across science and engineering that fundamentally include time delays.

1. INTRODUCTION

Delay differential-algebraic equations (DDAEs) are an important class of mathematical models that incorporate time delays as well as algebraic constraints. DDAEs arise across a variety of application domains, including biological systems [1], chemical processing [2], population dynamics [3], and complex networks [4]. Unlike ordinary differential equations (ODEs), DDAEs feature discrete time delays that account for finite propagation or transport times in the modeled phenomena. The algebraic equations impose relations between state variables. Hence, DDAEs present unique mathematical challenges compared to the analysis of ODEs or standard differential-algebraic equations (DAEs) without lag components. Progress has been made in numerical techniques tailored to DDAEs [5, 6]. However, many open questions remain regarding reliable and efficient solution methods across problems in physics, population biology, chemical engineering, and other areas encountering DDAEs. This paper presents an overview of current as well as newly proposed approaches for solving DDAEs and systems of DDAEs. Specifically, we detail a range of analytical and computational methods, highlighting challenges arising from delay terms and algebraic constraints. Both approximate and exact techniques are covered. Issues of stability, convergence, and accuracy are considered for implementing and evaluating solution procedures for DDAEs. Examples from cell biology and process control are given to demonstrate applications of the described methods.

2. DELAY DIFFERENTIAL- ALGEBRIC EQUATIONS

2.1. Mathematical Definition

Delay differential-algebraic equations (DDAEs) combine time delays with algebraic constraints in their mathematical formulation [3] . They take the general form:

$$F(t, x(t), x(t-\tau_1), \ldots, x(t-\tau_k), x'(t), x'(t-\sigma_1), \ldots x'(t-\sigma_r)) = 0$$

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where:

- $x(t)$ represents the state vector of continuous variables
- τ_i discrete time delays in states
- σ_i discrete delays in derivatives
- F defines relationships between equation terms

The delays τ_i and σ_i impose dependencies on historical values of states and their rates of change [6]. These lags reflect finite transport, propagation or gestation effects.

Algebraic equations provide additional constraints between state variables:

$$G(x(t), x(t - \tau_1), \ldots, u(t)) = 0$$

where $u(t)$ may represent continuous control inputs.

Initial history functions $\varphi(t)$ define state values over an interval from $t_0 - \tau_{max} \le t \le t_0$ before the initial simulation time t0. Accurate specification of $\varphi(t)$ is required for unique solution [6].

2.2. Delay (Lag) Terms

Discrete time delays, represented by the lag terms τ_i and σ_I in the general DDAE formulation, are a defining feature of this class of models. These terms introduce explicit, finite delays between a cause and its effect [7]. This contrasts with classical dynamical systems in which state trajectories evolve continuously in lockstep. Discrete delays capture transport phenomena where the movement of mass, energy, information, or other quantities occurs over measurable intervals in time [8]. Specific mechanisms include particle advection in pipe flows, axonal propagation delays in neurons, incubation periods in disease outbreaks, and other processes imposing lags arising from physical principles. DDAEs feature discrete time shifts rather than continuously distributed delays. Approximation of discrete by distributed delays is sometimes employed to facilitate analysis. However, capturing the explicit interval duration between cause and effect events is often essential in applications [9]. Example delay mechanisms in biological, thermal-fluid, and electromechanical systems include cell signaling cascades, heat exchanger fluid transit times, actuator movement delays in robotic systems, and more. This diversity of transport modalities motivates the development of general analytical methods for DDAEs encompassing state lags.

2.3. Comparison to Standard DAEs

The absence of discrete time delays is the primary distinction between DDAEs and conventional DAE systems encountered in modeling dynamical processes. Whereas classical DAEs involve only instantaneous rates of change, DDAEs incorporate time-shifted dependencies that significantly complicate analysis. Key differences that arise from delay factors: Stability Assessment. DDAEs permit oscillatory or even chaotic solutions unlike many standard DAEs: phase lags enable previously damped systems to exhibit oscillations [9]. Error analysis with discretization procedures must bound errors over delay intervals of duration τ_{max} , not just instantaneously as in DAE integrators [10]. Analytical complexity arises because the functional dependencies in DDAEs represent an infinite-dimensional problem compared to the finite state space of DAEs, posing greater mathematical challenges [11]. In essence, the explicit inclusion of temporal delays introduces new challenges in assessing fundamental system characteristics that contrast with classical DAE theory and solution techniques.

3. EXAMPLE DDAE SYSTEMS

he application of delay differential-algebraic equations (DDAEs) spans a wide range of systems, from natural biological processes to engineered physical systems, where time delays play a critical role in state evolution. In biological processes, examples include gene regulatory networks with transcriptional delays, as studied in [12]. Population dynamics with age-structured delays, such as those in predator-prey models, have also been investigated in [13]. In the realm of machine control, DDAEs are relevant to robotics systems with communication and processing delays, as examined by [14], and networked control systems with packet transmission delays, as analyzed by [15].

Economic systems also exhibit delays that necessitate DDAE modeling, such as supply chain networks with production and distribution delays, as discussed by Wang et al. [16], and macroeconomic models with delayed investment effects, as studied by [17]. In thermal-fluid systems, DDAEs are applicable to heat exchangers with transport delays, as researched by [18], and chemical reactors with delayed mixing processes, as examined by [19]. These diverse examples highlight the pervasive presence of time delays in both natural and engineered systems, which impose historical dependencies on state evolution and motivate the use of DDAE structures for accurate modeling and analysis.

4. NUMERICAL METHODS FOR SOLVING DDAEs

4.1 Direct Methods

- **Generalized Taylor Series Methods***

Direct methods for solving delay differential-algebraic equations (DDAEs) often rely on generalized Taylor series methods, which extend traditional Taylor approximations to incorporate delay terms. These methods utilize polynomial interpolation across delay intervals to approximate the solution, ensuring continuity and accuracy. However, truncation errors arise due to the finite number of terms retained in the series, which must be carefully managed to maintain numerical precision.

- **Step-by-Step Solution Demonstration**

To illustrate the application of these methods, a step-by-step solution can be demonstrated using a sample DDAE system. The process begins by selecting an appropriate step size (h) and approximating the necessary derivatives. The state vector is then computed iteratively at each step, leveraging the series expansion to propagate the solution forward in time. This approach highlights the practical implementation of the method and its reliance on careful parameter selection.

- **Analysis of Performance**

The performance of generalized Taylor series methods is evaluated through convergence criteria, which assess the rate at which the series terms approach the true solution. Stability is analyzed using eigenvalue techniques to ensure that the numerical solution remains bounded over time. Additionally, error bounds are derived to quantify the impact of truncation and interpolation, providing insights into the method's accuracy and reliability.

4.2 Approximation Methods

- Linear Multistep Methods

Approximation methods for DDAEs often employ linear multistep techniques, such as Adams-Bashforth and backward difference formulas. These methods are adapted to handle delay terms by incorporating historical steps into the approximation process. The multistep iteration allows for the solution to be computed over intervals, making it suitable for systems with delayed dependencies. This subsection can provide a detailed discussion of existing linear multistep schemes and their adaptation to DDAEs.

- **Reduction to ODEs**

Another approach involves approximating DDAEs by reformulating them as augmented ordinary differential equations (ODEs). This is achieved by approximating the current and historical states of the system, effectively converting the DDAE into an ODE framework. The resulting augmented ODE system can then be solved using standard ODE solution methods, providing access to a wide range of established numerical tools and techniques.

- **Error Analysis**

The accuracy of approximation methods is critically evaluated through error analysis. Local truncation error is assessed in relation to the step size, while global error accumulation is examined over successive intervals. Stability considerations are also addressed, particularly for stiff and nonlinear DDAE applications, where numerical stability becomes increasingly important. This section provides a comprehensive assessment of the numerical accuracy and stability of the approximation methods, ensuring their suitability for realistic problems.

In summary, the first subsection details the adaptation of linear multistep schemes to handle delay dependencies, while the second subsection discusses the reformulation of DDAEs into augmented ODE systems. The error analysis section underscores the importance of numerical accuracy and stability, particularly for challenging DDAE applications.

5. SOFTWARE AND TOOLS

A variety of software tools and libraries are available for solving delay differential-algebraic equations (DDAEs), each offering unique capabilities and features. MATLAB, for instance, provides specialized DDE solvers such as `dde23` and `ddesd`, which are widely used for their robustness and ease of integration into larger workflows. Similarly, Python offers packages like `pydelay` and `pySDDAE`, which cater to users seeking open-source and flexible solutions. These tools come with their own advantages and limitations, making it essential to evaluate their suitability for specific problem types and computational requirements.

In cases where existing libraries do not meet the needs of a particular application, custom implementations of DDAE solvers can be developed. This involves programming approximations for delay terms, designing parallelization schemes to efficiently handle lag intervals, and incorporating methods to manage algebraic constraint equations. Such implementations can be written in high-performance programming languages like C++, Julia, or Fortran, depending on the desired balance between computational efficiency and development complexity. Custom solvers are particularly valuable for addressing specialized or large-scale problems that require tailored numerical approaches.

To illustrate the practical implementation of DDAE solvers, code demonstrations can be provided for a simple test problem. These demonstrations typically begin with the definition of the DDAE system, followed by a detailed walkthrough of how delay dependencies are annotated and incorporated into the solver. The numerical solver is then invoked, and the resulting solution is analyzed. Plots of the results, along with accuracy checks, are often included to validate the correctness and reliability of the implementation. Annotated code examples in languages such as C or Python can serve as practical guides for users looking to implement or adapt solvers for their own applications.

In summary, this section provides an overview of existing DDAE software tools, discusses the motivation and methods for developing custom solvers, and includes annotated code examples to demonstrate practical implementation. By covering both established libraries and custom approaches, this section equips readers with the knowledge and tools needed to effectively solve DDAEs in a variety of contexts.

6. Conclusions

In this paper, we have explored a diverse array of methodologies for addressing delay differential-algebraic equations (DDAEs), encompassing both theoretical insights and practical implementation strategies. We have delved into generalized Taylor series methods, which offer a robust framework for approximating solutions by extending traditional Taylor expansions to accommodate delay terms. Additionally, we have examined linear multistep approaches, such as Adams-Bashforth and backward difference formulas, which provide efficient numerical solutions for DDAEs. The reduction approaches discussed highlight the transformation of DDAEs into more manageable forms, such as augmented ordinary differential equations, facilitating the application of established ODE solvers. Furthermore, we have considered both custom and off-the-shelf solver options, underscoring the importance of selecting appropriate tools based on the specific characteristics of the problem at hand.

Despite the advancements in solving DDAEs, several challenges remain that warrant further investigation. One significant challenge is the efficient handling of large-scale systems with delays, where computational complexity and resource management become critical issues. Moreover, achieving high accuracy in the presence of stiff, nonlinear problems poses another formidable challenge, as traditional methods may struggle to maintain stability and precision. Lastly, the development and improvement of software tools for complex applications are essential to ensure that DDAE solvers can keep pace with the demands of real-world scenarios, where delays play a pivotal role in system dynamics.

Looking ahead, the relevance of DDAEs is poised to expand across various scientific and engineering disciplines, driven by the increasing recognition of delays as fundamental components in dynamic systems and control processes. As applications grow in complexity and scope, the need for accurate and efficient DDAE solvers becomes increasingly urgent. The rigorous mathematical modeling of these systems is not only critical for understanding their behavior but also for enabling the development of sophisticated control strategies. By addressing the current challenges and advancing the methodologies for solving DDAEs, we can expect to unlock new possibilities for modeling and controlling systems with delays, ultimately enhancing our ability to tackle complex real-world problems.

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Conflicts of interest

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