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Research Article

Numerical Solution of Burgers'-Type Equations Using Modified Variational Iteration Method

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ABSTRACT

In this paper, we present modified variational iteration algorithm-I for the numerical solution of Burger's equations. In this recently developed technique, an auxiliary parameter is introduced which speed up the convergence rate of the series solutions. The suggested technique gives approximate and exact solutions with easily computable terms to linear and nonlinear PDEs without the Adomian polynomials, small perturbation, discretization or linearization. In order to assess the precision, reliability and compactness, results of the proposed procedure are compared with the variational iteration method, which reveals that the MVIA-I exceptionally productive, computationally attractive and has more accuracy than the others methods. Two numerical test problems are given to judge the behaviour of the modified algorithms and absolute errors are used to evaluate the accuracy of the method. Numerical results are carried out for different values of the parameters.

1. INTRODUCTION

Partial differential equations are extensively utilized in engineering and applied sciences to model a wide range of physical phenomena. Solving these equations is essential for understanding the physical behavior exhibited by these processes. While exact or analytical solutions for these partial differential equations are preferred, they are only attainable for straightforward problems with well-defined boundary condition. This limitation arises from mathematical complexities and the substantial space-time resources required. Acquiring precise solutions for non-linear partial differential equation came across in science and engineering can be challenging and inefficient. Hence, approximate or numerical methods continue to be crucial options for addressing these problems computationally. Numerical solutions for partial differential equations have generated significant relevance and have been focus of growing research attention in recent times.

The variational iteration method (VIM), as outlined in [1], has gained popularity in numerical approximations in recent years. This method eliminates the need for domain discretization and linarization of the specified differential equations. To find the analytical solution in sequence form, we calculate the Lagrange multiplier [2] by constraining the nonlinear terms within the given differential equation. Compared to other methods in the literatures, the variational iteration method is more versatile and can be executed with maximum efficiency.

For the majority of partial differential equations (PDEe), as the precise solution are unavailable, the alternative is to obtain numerical solutions for such problems. Our objective is to devise and employ a numerical procedure for solving the non-linear Burger's equation [3], that surpasses existing approaches as for convergence,accuracy and computational effectiveness

In this paper, we employ the variation Iteration method (VIM) including with an the unknown auxiliary parameter (h)

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for the numerical solution of non-linear Burgers'-type equations. Burger-type equation are a class of partial differential equation that emerge in a variety of contexts of applied mathematics, particularly in fluid dynamics, combustion, and nonlinear wave propagation [4]. The general form of a one-dimensional burger-type equation [5] are given by:

$$\frac{\partial y}{\partial t} + y\frac{\partial y}{\partial x} = \upsilon\frac{\partial^2 y}{\partial x^2} \tag{1}$$

where 'y' is the dependent variable representing the velocity , 't' is time, x is space, and υ is a small parameter representing the viscosity or diffusion coefficient. These equations exhibit nonlinear behavior and are known for their ability to model shock waves and other complex phenomena. Burgers' equation [6, 7], a nonlinear PDE, occurs in several engineering and scientific applications.

The standard Burgers equation, belongs to the class of non-linear partial differential equations, has garnered significant attention from researchers exploring diverse physical phenomenon such as fluid dynamics, shock wave theory, and turbulent flow [8–15]. This equation is more significance workable formulation of the behaviour of the shock waves in which non-linear diffusion and advection can be noted, [16].

The Burger's equation is first assessed by Bateman's, who develop the steady state solution for it and Burger's presented it as the turbulent flow for mathematical model. Cole and Hope individually demonstrated subsequently that it can convertible into linear heat equations [17]. In recent times, the Burger's equation continues to draws the researchers attention. As a result of the inclusion of an advection term yy_x and a viscosity term νy_{xx}, it serves as a model for testing other numerical techniques. Actually, the Burger's equation represent one dimensional Navier-stokes equation when the force and pressure term are removed from the Navier–stokes formula, [18]. An additional significance of these equation is to provide us to compares the qualities of numerical strategies applied to a non-linear equation.

In many areas of science and engineering, nonlinear PDEs are frequently employed to model and analyse nonlinear processes. The employment of numerical approaches continuously remains a significant alternative for the solution of PDEs because the majority of PDEs lack a general closed-form exact solution. For the solutions of partial differential equation (PDEs), numerous numerical technique, such as splines, spectral techniques, finite element techniques, and finite difference techniques [19], are developed. However, numerical approaches have certain drawbacks that make them difficult to use in complicated geometries, such as mesh formation, sluggish convergence, spatial dependence, stability, and low accuracy. Therefore, practical methods for solving these kinds of rather difficult equations must include efficient and accurate numerical approaches.

In order to obtain precise solutions for the various classes of these PDEs, the goal of this research paper is to present an unknown parameter in the variational iteration approach.

In recent times, numerous techniques have emerged for addressing both linear and non-linear problems. These include the variation iteration method (VIM) [20], the homotopy Analysis method [21], and the adomian decomposition method (ADM) [22], the Reduced Differential Transform method (RDTM) [23], modified variational iteration algorithms [24, 25], Reproducing kernel method [26], local meshless method [27, 28] and other homotopy perturbation method (HPM) [29]. The VIM is demonstrates its effectiveness in solving diverse linear or nonlinear differential equations, exhibiting minimal sensitivity to the degree of the non-linear terms. Furthermore, it simplifies calculations, reducing their complexity.

In this paper, we utilize the Modified variational iteration method [30], to find the exact solutions for several types of Burger's nonlinear partial differential equations.

2. MODIFIED VARIATIONAL ITERATION ALGORITHM-I (MVIA-I)

To illustrate the general concept of modified variational iteration algorithm-I, consider the following nonlinear differential equation

$$L[y(x)] + N[y(x)] = c(x), \tag{2}$$

where, $L[y(x)]$ represent the linear term, $N[y(x)]$ denotes the nonlinear term, and $c(x)$ represent the source term. The approximately solution $y_{k+1}(x)$, of the equation (2) with the initial condition $y_0(x)$, can be found by using the following correctional function:

$$y_{k+1}(x) = y_k(x) + \int_0^x \lambda(\psi)\left[L\{y_k(\psi)\} + N\{y_k(\psi)\} - c(\psi)\right] d\psi, \tag{3}$$

where λ, is the parameter called the Lagrange multiplier [31], which can determine by applying 'δ' to both sides of the recurrence relation (3) w,r,t the variable $y_k(x)$, resulting in:

$$\delta y_{k+1}(x) = \delta y_k(x) + h\delta\int_0^x \lambda(\psi)\left[L\{y_k(\psi)\} + N\{y_k(\psi)\} - c(\psi)\right] d\psi, \tag{4}$$

here $y_k(\psi)$, is restricted term, i.e., $\delta y_k(\psi) = 0$, and gives the following Lagrange multipliers:
$\lambda = -1$ for $m = 1$,
$\lambda = s - t$ for $m = 2$.
Moreover, general formula for obtaining the Lagrange multiplier in the cases of $m \geq 1$ as follows:

$$\lambda = \frac{(-1)^m (s-t)^{m-1}}{(m-1)!}. \tag{5}$$

After that, to find the value of λ, the iteration formula is formed by putting this value in the correction functional (3) as

$$y_{k+1}(x) = y_k(x) + \int_0^x \frac{(-1)^m (s-t)^{m-1}}{(m-1)!}) \left[L\{y_k(\psi)\} + N\{y_k(\psi)\} - c(\psi) \right] d\psi, \tag{6}$$

The iterative sequence is derived by starting with an appropriate the initial approximation and applying the iterative formula (6) successively. Repeated iterations are advantageous for attaining the required accuracy in the modern computer method. The Exact solution $y(x)$ is received as follow:

$$y(x) = \lim_{k \to \infty} y_k(x). \tag{7}$$

Finally in conclusion, the entire process of solving equation (2) can be expressed in iterative terms as:

$$\begin{cases} y_0(x) \text{ is a proper initial approximation} \\ y_1(x) = y_0(x) + \int\limits_0^x \lambda(\psi)[L\{y_0(\psi)\} + N\{y_0(\psi)\} - c(\psi)]d\psi \\ y_{k+1}(x) = y_k(x) + \int\limits_0^x \lambda(\psi)[L\{y_k(\psi)\} + N\{y_k(\psi)\} - c(\psi)]d\psi \\ n = 1, 2..., \end{cases} \tag{8}$$

The algorithm outlined in (8) for deriving approximate solutions is provided to as the variation iteration algorithm-I (VIA-I) [25]. It's an advanced iteration technique based on the widely employed lagrange multiplier principle [32]. In recent times, this approach has been applied to provide numerical and analytical solution for an expanded range of problems in different field of engineering [33] and applied science. The primary objective of this paper is to provide the auxiliary convergence term, denoted as "h," into the iteration formula. This term is to employed to assess the velocity of the approximate solution. In such manner, Algorithm (8) requires the most improve form, given below:

$$\begin{cases} y_0(x), \text{ is a proper initial estimation} \\ y_1(x,h) = y_0(x) + h\int\limits_0^x \lambda(\psi)[L\{y_0(\psi)\} + N\{y_0(\psi)\} - c(\psi)]d\psi \\ y_{k+1}(x,h) = y_k(x,h) + h\int\limits_0^x \lambda(\psi)[L\{y_k(\psi,h)\} + N\{y_k(\psi,h)\} - c(\psi,h)]d\psi \\ n = 1, 2..., \end{cases} \tag{9}$$

This method is known as modified variational iteration algorithm-I, which eliminates the need for domain discretization and the linarization of the provided differential equations. To find the numerical or analytical solution of the given differential equation, we must compute the Lagrange multiplier by constraining nonlinear terms. This process allows us to express the solution in a sequence form. When the auxiliary parameter $h = 1$, the variation iteration algorithm defined in equation (9) reverts to the standard (VIA-I). With this configuration, the approximate solution converges onto the exact solution as the iteration when 'n' tends to infinity. It's clear that the modified variation iteration algorithm is straightforward to implement. In nonlinear problems, the nonlinear terms must be regarded as restrained variations to obtain the value of the Lagrange multipliers. Subsequently, the recurrence relationship can be readily founded by substituting its determined value into the correction functional [34].

3. NUMERICAL EXAMPLES

In this section, we discuss two test problems of different types of Burger's equations to check the accuracy of the proposed method. We assess the accuracy of the method by taking different values of auxiliary parameter, and the obtained results are very encouraging and significant, while the absolute errors are calculated and compared with the error of the other methods available in the literature. For numerical computation, MATLAB R2016a are used on a Dell core i5,7G with 4GB of RAM.

3.1 Test problem 1

Consider the following Burger's equation

$$\frac{\partial y}{\partial t} + y\frac{\partial y}{\partial x} - \frac{\partial^2 y}{\partial x^2} = 0, \tag{10}$$

and the initial condition

$$y(x,0) = x, \tag{11}$$

the exact solution of equation (10) is given by [**?**],

$$y(x,t) = \frac{x}{1+t}. \tag{12}$$

In order to find the solution of the above Burger's equation by (MVIA-I), We can construct the correctional function for the equation (10), which is:

$$\begin{aligned} y_{n+1}(x,t,h) = y_n(x,t,h) + h\int_0^t \lambda(\psi)[\frac{\partial y_n(x,\psi,h)}{\partial(\psi)} + y_n(x,\psi,h)\frac{\partial y_n(x,\psi,h)}{\partial(x)} \\ -\frac{\partial^2 y_n(x,\psi,h)}{\partial(x)^2}]d\psi. \end{aligned} \tag{13}$$

The value of $\lambda(\psi)$ can be obtained through the theory of variational principles [35]. We determine the value of $\lambda(\psi) = -1$. Substituting this value of $\lambda(\psi)$ into equation (13), yields the following result:

$$\begin{aligned} y_{n+1}(x,t,h) = y_n(x,t,h) - h\int_0^t [\frac{\partial y_n(x,\psi,h)}{\partial(\psi)} + y_n(x,\psi,h)\frac{\partial y_n(x,\psi,h)}{\partial(x)} \\ -\frac{\partial^2 y_n(x,\psi,h)}{\partial(x)^2}]d\psi. \end{aligned} \tag{14}$$

We start the process with an initial approximation $y_0 = y(x,0) = x$, provided by equation (11), in the iteration formula (14) mentioned above, we can obtained directly the successive approximations which is:

$$\begin{aligned} y_0(x,t) &= x. \\ y_1(x,t,h) &= x - h*t*x. \\ y_2(x,t,h) &= -(x*h^3*t^3)/3 + x*h^2*t^2 + x*h^2*t - 2*x*h*t + x. \\ y_3(x,t,h) &= -(x*h^7*t^7)/63 + (x*h^6*t^6)/9 + (2*x*h^6*t^5)/15 \\ &\quad - (7*x*h^5*t^5)/15 - (x*h^5*t^4)/2 - (x*h^5*t^3)/3 \\ &\quad + (7*x*h^4*t^4)/6 + (5*x*h^4*t^3)/3 - (7*x*h^3*t^3)/3 \\ &\quad - 2*x*h^3*t^2 - x*h^3*t + 3*x*h^2*t^2 + 3*x*h^2*t \\ &\quad - 3*x*h*t + x. \end{aligned}$$

We stop the process at the third-order approximation. In order, to determine the better values of auxiliary parameter h, we define the residual function for the approximate solution

$$r_3(x,\ t,\ h) = \frac{\partial y_3(x,\ t,\ h)}{\partial t} + y_3(x,\ t,\ h)\frac{\partial y_3(x,\ t,\ h)}{\partial x} - \frac{\partial^2 y_3(x,\ t,\ h)}{\partial x^2} = 0. \tag{15}$$

The above residual function minimal values occurs at different values of auxiliary parameter. The value of the auxiliary parameter 'h' is determined using Matlab mathematical software. Substituting the value of the auxiliary factor (h) into $y_3(x,t,h)$, the results decreases in the absolute errors of the 3rd-order approximation of the suggested algorithm. To shows the precision, efficiency and accuracy of our suggested algorithm, we compare our results with those obtained from [20, 23], which shows that our suggested algorithm is very beneficial than the variational iteration method [1, 20], and reduced differential transform method [23]. It can be shown from Tables (I), (II), (III) and (IV), that the modified variation iteration algorithm-I converges quickly and gives most accurate results for all values of different parameters. The graphs of different values for t and x shows the behaviour of the Burger's equation at different values, which can be seen in figure (1) and figure (2).

TABLE I. Comparison of the exact and numerical solution for Test problem 1 with a fixed value of x and different values of t.

x	t	h	Exact Solution	MVIM Solution	VIM [1]
0.1	0	-1	0.100000000000	0.100000000000	0.100000000000
0.1	0.1	0.97071	0.090909090909	0.090909322335	0.090906344285
0.1	0.2	0.94560	0.083333333333	0.083336134172	0.083296690793
0.1	0.3	0.92373	0.076923076923	0.076934004923	0.076766752857
0.1	0.4	0.90445	0.071428571428	0.071455606704	0.071008243809
0.1	0.5	0.88725	0.066666666666	0.066719014228	0.065786210317

TABLE II. Absolute Errors for Test problem 1 at different value of t and fixed values of x.

x	t	Absolute error in MVIM	Absolute error in VIM
0.1	0	0	0
0.1	0.1	2.314263961300522e-07	2.746623376626012e-06
0.1	0.2	2.800838739464018e-06	3.664253968255204e-05
0.1	0.3	1.092800067842292e-05	1.563240659340631e-04
0.1	0.4	2.703527612223089e-05	4.203276190476307e-04
0.1	0.5	5.234756210720226e-05	8.804563492063461e-04

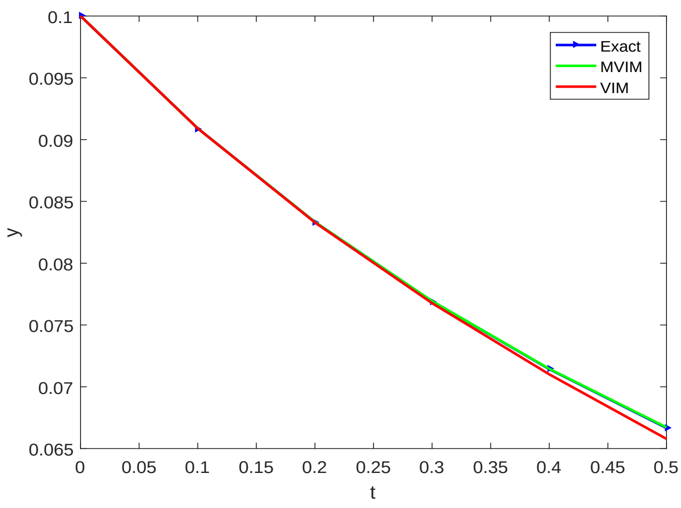

Fig. 1. Numerical result for Test problem 1 at different values of t and fixed value of x.

TABLE III. Comparison of the exact and numerical solutions for Test problem 1 at fixed value of t and different values of x.

t	x	h	Exact Solution	MVIM Solution	VIM [1]
0.1	0	-1	0	0	0
0.1	0.1	0.97071	0.090909090909	0.090909322335	0.090906344285
0.1	0.2	0.97072	0.181818181818	0.181818644670	0.181812688571
0.1	0.3	0.97073	0.272727272727	0.272727967006	0.272719032857
0.1	0.4	0.97074	0.363636363636	0.363637289341	0.363625377142
0.1	0.5	0.97075	0.454545454545	0.454546611677	0.454531721428

TABLE IV. Absolute errors for Test problem 1 at different values of x and fixed value of t.

t	x	Absolute Error in MVIM	Absolute Error in VIM
0.1	0	0	0
0.1	0.1	2.314263961300522e-07	2.746623376626012e-06
0.1	0.2	4.628527922601045e-07	5.493246753252024e-06
0.1	0.3	6.942791883624011e-07	8.239870129878035e-06
0.1	0.4	9.257055845202089e-07	1.098649350650405e-05
0.1	0.5	1.157131980622506e-06	1.373311688307455e-05

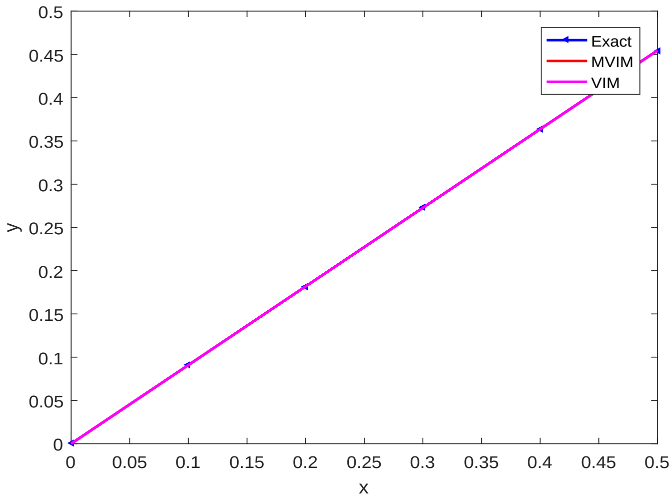

Fig. 2. Numerical result for Test problem 1 at different values of x and fixed value of t.

3.2 Test problem 2

Consider the following KDV Burger's equation

$$\frac{\partial y}{\partial t} + 2y\frac{\partial y}{\partial x} + 2y\frac{\partial^3 y}{\partial x^3} = 0, \tag{16}$$

and the initial condition

$$y(x, 0) = x, \tag{17}$$

the exact solution of equation (16) is given by [**?**],

$$y(x, t) = \frac{x}{1 + 2t}. \tag{18}$$

To solve the above KDV Burger's equation by MVIA-I, we can construct the correctional function for the equation (16), which is:

$$y_{n+1}(x,t,h) = y_n(x,t,h) + h\int_0^t \lambda(\psi)[\frac{\partial y_n(x,\psi,h)}{\partial(\psi)} + 2y_n(x,\psi,h)\frac{\partial y_n(x,\psi,h)}{\partial(x)} \\ +2y_n(x,\psi,h)\frac{\partial^3 y_n(x,\psi,h)}{\partial x^3}]d\psi. \tag{19}$$

The value of $\lambda(\psi)$ can be find through the theory of variational principles [35]. We determine the value of $\lambda(\psi)$, to be $\lambda(\psi) = -1$. Substituting this value of $\lambda(\psi)$ into equation (19), yields the following result:

$$y_{n+1}(x,t,h) = y_n(x,t,h) - h\int_0^t [\frac{\partial y_n(x,\psi,h)}{\partial(\psi)} + 2y_n(x,\psi,h)\frac{\partial y_n(x,\psi,h)}{\partial(x)} \\ +2y_n(x,\psi,h)\frac{\partial^3 y_n(x,\psi,h)}{\partial x^3}]d\psi. \tag{20}$$

We start the process with an initial approximation $y_0 = y(x,0) = x$, provided by equation (17), in the iteration formula (20) mentioned above, we can obtained directly the successive approximations, which is:

$$y_0(x,t) = x.$$
$$y_1(x,t,h) = x - 2*h*t*x.$$
$$y_2(x,t,h) = -(8*x*h^3*t^3)/3 + 4*x*h^2*t^2 + 2*x*h^2*t - 4*x*h*t + x.$$
$$\begin{aligned} y_3(x,t,h) = &-(128*x*h^7*t^7)/63 + (64*x*h^6*t^6)/9 + (64*x*h^6*t^5) \\ &-(224*x*h^5*t^5)/15 - 8*x*h^5*t^4 - (8*x*h^5*t^3)/3+ \\ &(56*x*h^4*t^4)/3 + (40*x*h^4*t^3)/3 - (56*x*h^3*t^3)/3- \\ &8*x*h^3*t^2 - 2*x*h^3*t + 12*x*h^2*t^2 + 6*x*h^2*t \\ &-6*x*h*t + x. \end{aligned}$$

We stop the process at the third-order approximation. To determines the better value of auxiliary parameter (h), we define the residual function for the approximate solution: as

$$r_3(x,\ t,\ h) = \frac{\partial y_3(x,\ t,\ h)}{\partial t} + 2y_3(x,\ t,\ h)\frac{\partial y_3(x,\ t,\ h)}{\partial x} \\ +2y_3(x,\ t,\ h)\frac{\partial^3 y_3(x,\ t,\ h)}{\partial x^3} = 0. \tag{21}$$

The above residual function minimal values occurs at different values of h. The value of the additional parameter 'h' is determined using Matlab mathematical software. Substituting the value of the auxiliary factor (h) into $y_3(x,t,h)$, the results is decreases in the absolute errors of the 3rd-order estimation of the suggested algorithm. To shows the accuracy of our suggested algorithm, we compare our results with those obtained from [1, 23], which show that our suggested algorithm is gives good results than the variational iteration method [1, 20], and reduced differential transform method [23]. It can be shown from Table (V), (VI), (VII) and (VIII), that the Modified variation iteration Algorithm-I converges quickly and gives most precise results for all values of various parameters. The graphs of distinct values for t and x shows the behaviour of the KDV Burger's equation at different values, which can be seen in figure (3) and figure (4).

TABLE V. Comparison of the numerical and exact results for Test problem 2 at fixed value of x and different values of t.

x	t	h	Exact Solution	MVIM Solution	VIM [1]
0.1	0	-1	0.100000000000	0.100000000000	0.100000000000
0.1	0.1	0.94560	0.083333333333	0.083336134172	0.083296690793
0.1	0.2	0.90445	0.071428571428	0.071455606704	0.071008243809
0.1	0.3	0.87176	0.062500000000	0.062587038925	0.060921965714
0.1	0.4	0.84473	0.055555555555	0.055737250052	0.051763829841
0.1	0.5	0.82154	0.050000000000	0.050302077899	0.042857142857

TABLE VI. Absolute errors for Test problem 2 at different values of t and fixed values of x.

x	t	Absolute error in MVIM	Absolute error in VIM
0.1	0	0	0
0.1	0.1	2.800838739464018e-06	3.664253968255204e-05
0.1	0.2	2.703527612223089e-05	4.203276190476307e-04
0.1	0.3	8.703892514029532e-05	0.001578034285714000
0.1	0.4	1.816944965975439e-04	0.003791725714286000
0.1	0.5	3.020778990240475e-04	0.007142857142857000

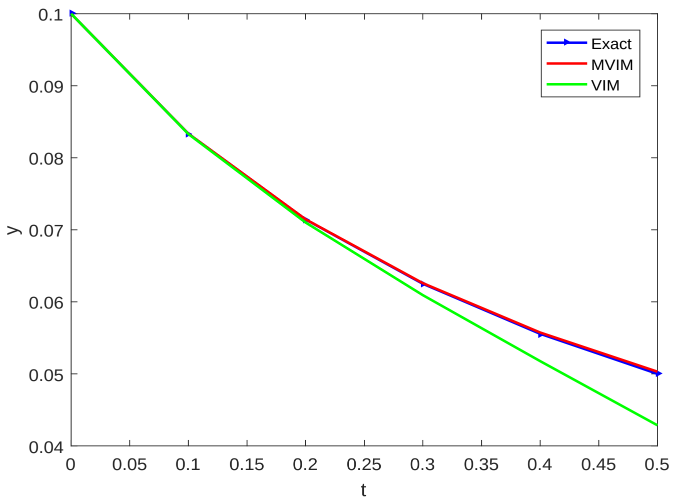

Fig. 3. Numerical results for Test problem 2 at different values of t and fixed values of x.

TABLE VII. Comparison of the numerical and exact solutions for test problem 2 at fixed value of t and different values of x.

t	x	h	Exact Solution	MVIM Solution	VIM [1]
0.1	0	-1	0	0	0
0.1	0.1	0.94560	0.083333333333	0.083336134172	0.083296690793
0.1	0.2	0.94561	0.166666666666	0.166672268344	0.166593381587
0.1	0.3	0.94562	0.250000000000	0.250008402516	0.249890072380
0.1	0.4	0.94563	0.333333333333	0.333344536688	0.333186763174
0.1	0.5	0.94564	0.416666666666	0.416680670860	0.416483453968

TABLE VIII. Absolute errors for test problem 2 at different value of x and fixed value of t.

t	x	Absolute error in MVIM	Absolute error in VIM
0.1	0	0	0
0.1	0.1	2.800838739464018e-06	3.664253968255204e-05
0.1	0.2	5.601677478928036e-06	7.328507936510409e-05
0.1	0.3	8.402516218364298e-06	1.099276190476284e-04
0.1	0.4	1.120335495785607e-05	1.465701587302082e-04
0.1	0.5	1.400419369729233e-05	1.832126984127047e-04

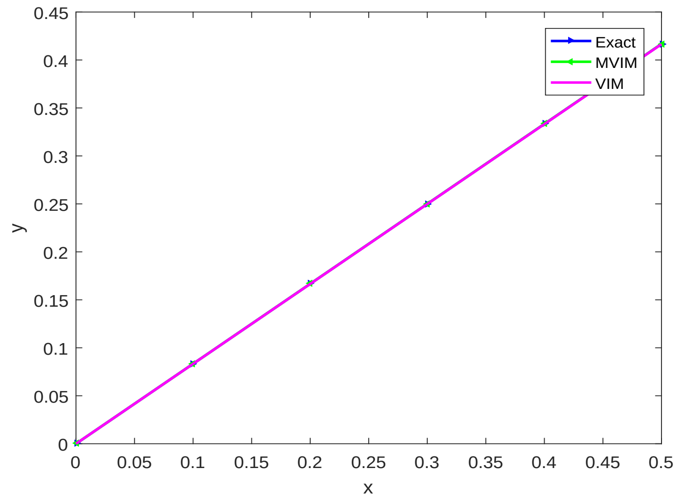

Fig. 4. Numerical results for Test problem.2 at different values of x and fixed values of t.

4. CONCLUSION

In this paper, we used the modified variational iteration algorithm-I for the numerical solution of Burger's equations. The technique effectively gives extremely precise solutions using various values of auxiliary parameter. This modified algorithm makes easy the computational work for solving linear and nonlinear problems arises in science and engineering, and results of high degree accuracy can be obtained in a few iterations as compared to earlier methods. The modified algorithm can be utilized without any need for discretization, linearization and lengthy calculations. We conclude that the proposed algorithm provides an accurate numerical/analytical solutions and can handle the nonlinear PDEs in a good and reliable manner in various cases.

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