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Research Article

Some Results on Commutativity for Alternative Rings With 2, 3-Torsion Free

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ABSTRACT

In this article, we establish and proof some theorem on commutativity of alternative ring with 2, 3 – torsion free satisfy the followi-*ng properties (Identities):

(p_1) $[x^2y^2 + y^2x^2, x] = 0$
(p_2) $[x(xy)^2 + (xy)^2x, x] = 0$
(p_3) $[x(x^2y^2), x] = 0$
(p_4) $[x(xy), x] = 0$ $\quad$ $for\ every\ x, y\ in\ R.$

1. INTRODUCTION

In this paper (article), we first study some result on commutativity of alternative rings with 2, 3-torsion free with some properties (constrain) that commute with (x). R represents an alternative ring, The Centre $Z(R) = [x \in R/xy = yx]$,The commutator $[x, y] = xy - yx$, the anti commutator, $x \circ y = xy + yx$, also $A(R)$ the assosymetric ring, $N(R)$ the set of nilpotent element.

An alternative ring R is a ring in which $(xx)y = x(xy), y(xx) = (yx)x$ for all x, y in R, these equations are known as left and right alternative laws respectively. An assosymetric ring A(R) is one in which $(x, y, z) = \big(p(x), p(y), p(z)\big)$, where p is any permutation of $x, y, z \in R$. An associator (x, y, z) we mean by $(x, y, z) = (xy)z - x(yz)$for all $x, y, z \in$ R. A ring R is called a prime if whenever A and B are ideals of R such that $AB = \{0\}$ then either $A = \{0\}$ or $B = \{0\}$. If in a ring R, the identity $(x, y, x) = 0$ i.e. $(xy)x = x(yx)$ for all x, y in R holds then R is called flexible. A ring R is said to be m-torsion free if $mx = 0$ implies $x = 0, m$ is any positive number for all $x \in R$.A non-associative rings R is an additive abelian group in which multiplication is defined, which is distributive over addition on left as well as on right $[(x + y)z = xz + yz,\ z(x + y) = zx + zy, \forall\, x, y, z \in R]$.

Abuja bal and Khan [1] proved the commutativity of associative ring satisfies the identity $(xy)^2 = xy^2x$.Gupta [2] established that a division ring R is commutative if and only if $[xy, yx] = 0$.

In addition, Madana and Reddy [3] have established the commutativity of non-associative ring satisfying the identities $(xy)^2 = x^2y^2$ and $(xy)^2 \in Z(R)\forall x, y \in R$.Further, Madana Mohana Reddy and Shobha lath. [4]Established the commutativity of non-associative primitive rings satisfying the identities:

$x(x^2 + y^2)$+$(x^2 + y^2)x \in Z(R)$and $x(xy)^2 - (xy)^2x \in Z(R)$, Modification by these Scrutiny(observation) it is exist natural to look commutativity of alternative rings satisfies: (p_1) ,(p_2), (p_3)& (p_4).

In the present paper we consider the following theorems.

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2. THE MAIN THEOREMS

Now, we begin with the proof of our theorems.

Theorem 1: *Let R be 2-torsion free alternative rings with unity satisfy the following constrain* (p_1) $\ for\ every\ x, y\ in\ R,$ *then R is commutative.*

Proof

$[x^2y^2 + y^2x^2, x]$

$x(x^2y^2 + y^2x^2) - (x^2y^2 + y^2x^2)x = 0$

$x(x^2y^2 + y^2x^2) = (x^2y^2 + y^2x^2)x$ (1)

Put $x = (x+1) in\ 1\ above$

$=> (x+1)[(x+1)^2y^2 + y^2(x+1)^2] = [(x+1)^2y^2 + y^2(x+1)^2](x+1)$

$=> (x+1)[(x^2+2x+1)y^2 + y^2(x^2+2x+1)] = [(x^2+2x+1)y^2 + y^2(x^2+2x+1)](x+1)$

$=> (x+1)[(x^2y^2 + 2xy^2 + y^2) + (y^2x^2 + 2y^2x + y^2)] = [(x^2y^2 + 2xy^2 + y^2) + (y^2x^2 + 2y^2x + y^2)](x+1).$

$=> x(x^2y^2) + x(2xy^2) + xy^2 + x(y^2x^2) + x(2y^2x) + xy^2 + y^2x^2 + 2xy^2 + y^2 + y^2x^2 + 2y^2x + y^2 = (x^2y^2)x + (2xy^2)x + y^2x + (y^2x^2)x + (2y^2x)x + y^2x + (x^2y^2) + 2xy^2 + y^2 + (y^2x^2) + 2y^2x + y^2.$

$=> x(x^2y^2 + y^2x^2) + x(2xy^2 + 2y^2x) + 2xy^2 + x^2y^2 + 2xy^2 + y^2x^2 + 2y^2x + 2y^2$

$= (x^2y^2 + y^2x^{2)})x + (2xy^2 + 2y^2x)x + 2y^2x + x^2y^2 + 2xy^2 + y^2x^2 + 2y^2x + y^2$

Using 1 above and collecting like terms we get

$=>\ x(2xy^2 + 2y^2x) + xy^2 + xy^2 = (2xy^2 + 2y^2x)x + y^2x + y^2x$ (2)

Apply 2-torsion free in 2 we had

$xy^2 + xy^2 = y^2x + y^2x \quad \leftrightarrow \quad 2xy^2 = 2y^2x$

$xy^2 = y^2x$ (3)

Insert $y = y + 1\ in\ \ 3\ above$

$x(y+1)^2 = (y+1)^2x$

$=> x(y^2 + 2y + 1) = (y^2 + 2y + 1)x$

$xy^2 + 2xy + y = y^2x + 2yx + y$ Using 3 above and collecting like terms we obtain.

$$2xy = 2yx$$

$$2(xy - yx) = 0$$

$xy = yx$ Which is commutative.

Theorem 2: *Let R be 2, 3-torsion free alternative rings with unity 1, satisfy the following property* (p_2) $\ for\ every\ x, y\ in\ R,$ *then R is commutative.*

Proof:

From our hypothesis i.e

$[x(xy)^2 + (xy^2)x, x]$ Then we had

$x[x(xy)^2 + (xy)^2x] = [x(xy)^2 + (xy)^2x]\ x$

$x[x(x^2y^2) + (x^2y^2)x] = [x(x^2y^2) + (x^2y^2)x]x$ (4)

Put $x = (x+1)$ $in\ 4\ above$

$=>(x+1)[(x+1)(x+1)^2y^2) + (x+1)^2y^2)(x+1)] = [(x+1)(x+1)^2y^2) + (x+1)^2y^2)(x+1)](x+1)$

$=>(x+1)[(x+1)(x^2y^2 + 2xy^2 + y^2) + (x^2y^2 + 2xy^2 + y^2)(x+1)] = [(x+1)(x^2y^2 + 2xy^2 + y^2) + (x^2y^2 + 2xy^2 + y^2)(x+1)](x+1)$

$=>(x+1)[x(x^2y^2) + x(2xy^2) + xy^2 + x^2y^2 + 2xy^2 + y^2 + (x^2y^2)x + (2xy^2)x + y^2x + x^2y^2 + 2xy^2 + y^2] = [x(x^2y^2) + x(2xy^2) + xy^2 + x^2y^2 + 2xy^2 + y^2 + (x^2y^2)x + (2xy^2)x + y^2x + x^2y^2 + 2xy^2 + y^2]\ (x+1).$

$=> x(x(x^2y^2)) + x(x(2xy^2)) + x^2y^2 + x(x^2y^2) + 2x^2y^2 + xy^2 + x(x^2y^2)x + (2x^2y^2)x + x(y^2x) + x(x^2y^2) + 2x^2y^2 + xy^2 + x(x^2y^2) + x(2xy^2) + xy^2 + x^2y^2 + 2xy^2 + y^2 + (x^2y^2)x + (2xy^2)x + y^2x + x^2y^2 + 2xy^2 + y^2 = [x(x^2y^2)x + x(2xy^2)x + (xy^2)x + (x^2y^2)x + (2xy^2)x + y^2x + ((x^2y^2)x)x + ((2xy^2)x)x + y^2x^2 + (x^2y^2)x + (2xy^2)x + y^2x + x(x^2y^2) + x(2xy^2) + xy^2 + x^2y^2 + 2xy^2 + y^2 + (x^2y^2)x + (2xy^2)x + y^2x + x^2y^2 + 2xy^2 + y^2].$

$=> x[x(x^2y^2) + (x^2y^2)x] + x[x(2xy^2) + (2xy^2)x] + 2x(x^2y^2) + 3(x^2y^2) + 3x(2xy^2) + 3xy^2 + x(y^2x) + 2xy^2 + 2xy^2 + 2y^2 + (x^2y^2)x + (2xy^2)x + y^2x = [x(x^2y^2) + (x^2y^2)x]x + [x(2xy^2) + (2xy^2)x]x + 3(x^2y^2)x + (3xy^2)x + (3xy^2)x + 3y^2x + (y^2x)x + 2(x^2y^2) + x(x^2y^2) + 3xy^2 + xy^2 + 2y^2.$

Collecting terms, Using 4 and applied 2, 3 -torsion free we get:

$y^2x = xy^2$ (5)

$put\ y = (y+1)$ in 5 above

$(y+1)^2x = x(y+1)^2$

$(y^2 + 2y + 1)x = x(y^2 + 2y + 1)$

$y^2x + 2yx + x = xy^2 + 2xy + x$
Collect like term and used 5 we arrived at:
$2yx = 2xy \quad <=> \quad 2yx - 2xy = 0$
$2(yx + xy) = 0$
Equate both sides we had
$yx + xy = 0$
$yx = xy <=> [x, y]$ is commutative hence the proof of theorem 2.
Theorem 3: *Let R be 2-torsion free alternative rings with unity satisfy the following constrain* (p_3) $\ for\ every\ x, y\ in\ R$, *then R is commutative.*
Proof:
$[x(x^2y^2), x] = 0$ The hypothesis can be re-write as
$x[x(x^2y^2) - (x^2y^2)x]x = 0$
$x[x(x^2y^2)] = [(x^2y^2)x]x$ (6)
Insert $x = (x + 1)\ in\ \ 6\ above.$
$(x + 1)[(x + 1)(\ x + 1)^2\ y^2] = [(\ x + 1)^2\ y^2(x + 1)](x + 1).$
$=>(x + 1)[(x + 1)(x^2 + 2x + 1)y^2] = [(x^2 + 2x + 1)y^2(x + 1)](x + 1).$
$=>(x + 1)[(x + 1)(x^2y^2 + 2xy^2 + y^2)] = [(x^2y^2 + 2xy^2 + y^2)(x + 1)](x + 1).$
$=>(x + 1)[x(x^2y^2) + x(2xy^2) + xy^2 + x^2y^2 + 2xy^2 + y^2] = [(x^2y^2)x + (2xy^2)x + y^2x + x^2y^2 + 2xy^2 + y^2](x + 1).$
$=>x[x(x^2y^2)] + x(2x^2y^2) + x^2y^2 + x(x^2y^2) + 2x^2y^2 + xy^2 + x(x^2y^2) + x(2xy^2) + xy^2 + x^2y^2 + 2xy^2 + y^2] = [(x^2y^2)x]x + [(2x^2y^2)]x + (xy^2)x + (x^2y^2)x + (2xy^2)x + y^2x + (x^2y^2)x + (2xy^2)x + xy^2 + x^2y^2 + 2xy^2 + y^2.$
We collect like terms, Using 6 and apply 2-torsion free we get.
$xy^2 = y^2x$ (7)
$put\ y = (y + 1)\ in\ 7\ above$
$x(y + 1)^2 = (y + 1)^2x$
$x(y^2 + 2y + 1) = (y^2 + 2y + 1)x$
$(xy^2 + 2xy + x) = (y^2x + 2yx + x)$
Apply 7 and collect like terms
$2xy = 2yx \quad <=> \quad 2(xy - yx) = 0$
$xy = yx\ \ is\ commutative\ hence\ the\ proof\ of\ theorem\ 3.$
Theorem 4: *Let R be 2-torsion free alternative rings with unity satisfy the following constrain* p_4 $\ for\ every\ x, y\ in\ R$, *then R is commutative.*
Proof.
From our hypothesis
$[x(xy), x]$
$x[x(xy)] - [x(xy)]x = 0$
$x[x(xy)] = [x(xy)]x$ (8)
Insert $x = (x + 1)\ \ in\ above\ 8$
$(x + 1)[(x + 1)(xy + y)] = [(x + 1)(xy + y)](x + 1)$
$(x + 1)[x(xy) + xy + xy + y] = [x(xy) + xy + xy + y](x + 1)$
$=>x[x(xy)] + x(xy) + x(xy) + xy + x(xy) + xy + xy + y] = [x(xy)]x + (xy)x + (xy)x + yx + x(xy) + xy + xy + y]$
$=>x[x(xy)]\ + 2x(xy) + xy + x(xy) + xy + xy + y] = [x(xy)]x + 2(xy)x + yx + x(xy) + xy + xy + y]$
Using 8 and apply 2-torsion free we get.
$xy + x(xy) + xy = \ yx + x(xy) + xy$ (9)
By Colleting like terms in 9 we had
$xy = yx\ \ or\ [x, y]$. Hence the proved

Hence the completion of the proved, as we can seen from the above both the properties (constrains): (p_1, p_2, p_3 & p_4) Are commutative and satisfy the Identities either $(xx)y = x(xy)\ \ or\ y(xx) = (yx)x$. So R is an Alternative rings as we stated it above, Hence an alternative rings with Identity together with commutativity yields $(x, x, y) = 0 = (y, x, x)\ in\ complition.$

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