



Research Article

On commutativity of alternative rings with $[xy^n x \pm yx^n y, x] = 0$ Abubakar Salisu^{1,*} , Shu'aibu Salisu³ , Mustapha Mannir Gafai² ¹ Department of Transport, Planning and Management, Federal polytechnic Daura Katsina State. Nigeria.² Department of Mathematics and Statistics Umaru Musa Yar'adua University Katsina. Nigeria.³ Department of computer science federal polytechnic Daura Katsina state. Nigeria.

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ABSTRACT

Let R be a n -torsion free with identity 1, In this article we investigate and prove the commutativity of alternative ring of the property (p_1) , (p_2) and (p_3) under suitable constraints.

$$(p_1) \quad [xy^n x \pm yx^n y, x] = 0$$

$$(p_2) \quad [xy^n x \pm yx^n y, y] = 0$$

$$(p_3) \quad [x(xy)^2 + (xy)^2 x, x] = 0. \quad \forall x, y \in R.$$

1. INTRODUCTION

In this paper R represent an alternative ring with center $Z(R) = \{x \in R \mid xy = yx\}$, the commutator $[x, y] = xy - yx$ and anti-commutator $xy \circ xy = xy + yx$ for any pair element $x, y \in R$, for any positive integer n , an element $x \in R$ is said to be $n!$ -torsion free if and only if $nx = 0$ implies $x = 0$. The associator (x, y, z) is define by $(x, y, z) = (xy)z - x(yz)$ for all $x, y, z \in R$, this plays a key role in the study of non-associative rings. It can be viewed as a measure of the non-associativity of a ring. In terms of associator, a ring is called left alternative if $(x, x, y) = 0$ right alternative if $(y, x, x) = 0$ for all $x, y \in R$ and alternative if both condition hold. i.e. $(x, y, y) = 0$ and $(y, y, x) = 0$.

In [2] established that a division ring R is commutative if and only if $[xy, yx] = 0$. Also generalize Gupta's result which assert that a semi prime ring R in which $[xy, yx] = xy^2 x - yx^2 y \in Z(R)$ or $xy \circ xy = xy^2 x + yx^2 y \in Z(R)$ is necessary commutative, also [1] proved the commutativity of associative ring satisfies the identity $(xy)^2 = xy^2 x$. also in their paper proved the properties: $xy \circ xy = xy^2 x + yx^2 y \in Z(R)$ and $xy^n x \pm yx^n y \in Z(R)$, most be commutative In addition, [3] have established the commutativity of non-associative ring satisfying the identities $(xy)^2 = x^2 y^2$ and $(xy)^2 \in Z(R) \forall x, y \in R$.

Further, [4] established the commutativity of non-associative primitive rings satisfying the identities: $x(x^2 + y^2) + (x^2 + y^2)x \in Z(R)$ and $x(xy)^2 - (xy)^2 x \in Z(R)$. Recently [5] show that some results on commutativity of some 2-torsion free non associative rings with unity satisfy:

$(\alpha\beta)^2 - \alpha\beta \in Z(R)$ for all $\alpha\beta$ in R Motivated by this observation it is natural to look commutativity of alternative rings satisfies

Motivated from them we establish the commutativity of ring with condition $(p_1), (p_2)$ and commutativity of alternative rings with (p_3) with suitable constraint.

Main Result.

The following are the main Results.

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Theorem 2.1.

Suppose that a n -torsion free alternative ring R with identity 1 and there exist a positive integer n such that: $[xy^n x \pm yx^n y, x] = 0$ or $[xy^n x \pm yx^n y, y] = 0 \quad \forall x, y \in R$. Then R is commutative.

Proof.

By hypothesis, we have

$$[xy^n x \pm yx^n y, y] = 0. \text{ In this property we consider } [xy^n x + yx^n y, y] = 0$$

$$[yxy^n x - xy^n xy + y^2 x^n y - yx^n y^2] = 0$$

$$[y, (xy^n x)] + [y, (x^n y - yx^n y)] \Leftrightarrow [y, xy^n x] + y[x^n, y] = 0$$

Replace $y = y + 1$ in 2.1 above and applied n -torsion free we obtained

$$[x^n C_1 y_1 x, y] + [x^n C_2 y_2 x, y] + \dots + [x^n C_n y_n x, y]$$

$$x[{}^n C_1 y_1 + {}^n C_2 y_2 + \dots + {}^n C_n y_n]x = [x^n, y^2] + yx^n = 0$$

We used binomial expansion and by inserting $y = y + 1$ for n -times and using the previous we obtained identity in every stage in above we had

$$x[({}^{n-1}C_{n-1} {}^{n-1}C_{n-2} + \dots + {}^n C_n)x, y].$$

This gives,

$n! [x^2, y] = 0$ in view of n -torsion free condition, we get on the commutator $[x^2, y] = 0$ for $n > 2$ then R is commutative.

Remark 2.2: The following Corollary is an immediate consequence of our main result if we set $n = 2$.

Corollary 2.3

Let R be a 2-torsion free alternative ring with identity 1. If R has a property:

$$[xy^2 x \pm yx^2 y, x] = 0 \text{ or } [xy^2 x \pm yx^2 y, y] = 0 \quad \forall x, y \in R. \text{ then } R \text{ is commutative ring.}$$

Proof

Since $[xy \circ xy, y]$, now we consider $[xy^2 x - yx^2 y, y] = 0$.

$$[xy^2 x - yx^2 y, y] = 0$$

$$[yxy^2 x - xy^2 xy + yx^2 y^2 - y^2 x^2 y] = 0$$

$$[y, xy^2 x] + y[x^2, y] = 0 \quad 2.3$$

Replace $y = y + 1$ in 2.3 above and applied 2-torsion free we obtained

$$y^2 x^2 - x^2 y^2 = 0 \Leftrightarrow [y^2, x^2] = 0 \quad 2.4$$

Replace $y = y + 1$ in 2.4 above we had

$$yx^2 - x^2 y = 0 \Leftrightarrow [y, x^2] = 0 \quad 2.5$$

Replace $y = y + 1$ in 2.5 above and applied 2-torsion free we obtained

$$yx - xy = 0 \Leftrightarrow [y, x] = 0 \text{ or } yx = xy \text{ implies } R \text{ is commutative.}$$

Theorem 3.1

Let R be a 2,3-torsion free alternative ring with unity satisfy $[x(xy)^2 + (xy)^2 x, x] = 0$, Then R is commutative.

Proof

From the hypothesis above in 3.1

$$x[x(xy)^2 + (xy)^2 x] = [x(xy)^2 + (xy)^2 x]x, \quad \text{for all } x, y \in R. \quad 3.2$$

Substitute $x = (1 + x)$ in , apply 2,3 torsions free and use (1) we get

$$y^2 x = xy^2, \quad \text{for all } x, y \in R. \quad 3.3$$

Substitute $y = (y + 1)$ in (3.3) and Apply 2-torsion

This implies $xy = yx$ and R is commutative.

Since R is a commutative ring and satisfies the identities either $(xx)y = x(xy)$ or

$y(xx) = (yx)x$, so that R is an alternative ring. Hence an alternative ring R with identity together with commutativity yields $(x, x, y) = 0 = (y, x, x)$, which completes the proof.

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Conflicts of interest

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