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Research Article

Fuzzy Classical Two-Absorbing Second (Secondary) Sub-modules and Strongly Classical Two-Absorbing Second Sub-modules

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ABSTRACT

Fuzzy classical Two-Absorbing second sub-modules, Fuzzy classical Two-Absorbing secondary sub-modules and strongly classical Two-Absorbing second sub-modules are presented in this search under the names of comultiplication fuzzy modules, cocyclic fuzzy modules, and multiplication fuzzy modules, along with some basic traits and qualities of these concepts. The relationship between fuzzy classical Two-Absorbing second sub-modules, fuzzy classical Two-Absorbing secondary sub-modules, fuzzy strongly classical Two-Absorbing second sub-modules, and other fuzzy second sub-module concepts is also covered.

1. INTRODUCTION

T is a commutative ring with identity in this study, and W is a unitary T-module, or simply T-module. In 1965, Zadeh [14] introduced the idea of fuzzy sets. In 1971, Rosenfeld presented the idea of fuzzy groups [1]. In [2], Deniz S. et al. introduced the idea of a 2-absorbing fuzzy ideal, which is a prime fuzzy ideal generalization. A classical 2-absorbing sub-module was presented by Hojjat et al. [5] in 2015 as a generalization of a classical prime (which is itself quasi-prime). sub-module "A classical 2-absorbing sub-module is a proper sub-module N of a T-module W if each time $a, b, c \in R$ and $m \in M$ with $abcm \in N$, then $abm \in N$ or $acm \in N$ or $bcm \in N$ ". In 2019, Hatam and Wafaa conducted research and presented the idea of a traditional T-ABSO fuzzy sub-module [12].

This paper is divided into three pieces. The definition of fuzzy classical Two-Absorbing second sub-module, necessary qualities, various hypotheses, theorems, and examples are examined and presented in Section (1). We examine strongly classical two-absorbing second sub-module conceptions and their relationship to two-absorbing second sub-module concepts in section (2). As a counterpart notion of Two-Absorbing primary sub-modules, fuzzy classical Two-Absorbing secondary sub-modules are introduced in section (3), and some associated findings are obtained.

Note that: The notations fzy ideal, fzy sub-module, fzy singleton, fzy second and fzy module represent the fuzzy ideal, fuzzy sub-module, fuzzy singleton, fuzzy second and fuzzy module.

2. CONCEPT BASIC

Definition 2.1[14]**:** Let S be a non-empty set and L be an interval [0,1] of the real line (real number). A fzy set A in S (a fzy subset of S) is a function from S into L.

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Definition 2.2 [7]:Let $x_u: S \to L$ be a fzy set in S, where $x \in S,\ u \in L$, identify by
$x_u(y) = \begin{cases} u & if\ x = y \\ 0 & if\ x \neq y \end{cases}$, x_u is named fzy singleton in S.
If $x = 0$ and $u = 1$, then $0_1(y) = \begin{cases} 1 & if\ y = 0 \\ 0 & if\ y \neq 0 \end{cases}$

Definition 2.3[15] : The term "fzy ideal of T" refers to a fzy subset K of a ring T, if ∀ x, y ∈ T:
1- $K(x - y) \geq min\{K(x), K(y)\}$
2- $K(xy) \geq max\{K(x), K(y)\}$.

Definition 2.4 [15]: Let W be a T-module fzy set Y of W is called fzy module of a T-module W if
1- $Y(x - y) \geq min\{Y(x), Y(y)\}$, for all $x, y \in W$.
2- $Y(rx) \geq Y(x)$, for all $x \in W, r \in T$.
3- $Y(0) = 1$ (0 is the zero element of W).

Definition 2.5[8]: Consider two fzy modules of a T-module W, Y and P. P is named fzy sub-module of Y if $P \subseteq Y$.

Proposition 2.6[**10]:**Let P be fzy set of W. Then the level subset $P_u, \forall u \in L$ is a sub-module of W iff P is fzy sub-module of fzy module of a T-module W.

Definition 2.7[13]: Assume that fzy module Y has two fzy sub-modules, P and B. The fzy subset of T, represented by $(P{:}\,B)$, is the residual quotient of P and B, which is defined by:
$(P{:}\,B)(r) = sup\{v \in L{:}\, r_v.B \subseteq P\}$ for all $r \in T$.That $(P{:}\,B) = \{r_v{:}\, r_v.B \subseteq\ P\ ; r_v\ is\ a\ F.singleton\ of\ T\ \}$. If $B =< x_k >$, then $(P{:} < x_k >) = \{r_v{:}\, r_v.x_k \subseteq P\ ; r_v\ is\ fzy.singleton\ of\ T\}$.

Definition 2.8[8]: Assume that P is a proper fzy sub-module of Y. The definition of the fzy annihilator of P, represented by Fzy-annP, is: $(Fzy - annP)(r) = sup\{v{:}\ v \in L\ , r_vP \subseteq 0_1\}$, for all $r \in T$.
Note that: $\mathrm{Fzy} - \mathrm{ann}P = (0_1{:}\,P)$, hence $(Fzy - annY)_v \subseteq annY_v$, [4].

Proposition 2.9[8]:If Y is fzy module of a T-module W, then Fzy-annY is fzy ideal of T.

Definition 2.10[11]:An perfect fzy If Ĥ of a ring T is non-empty and for any a_s, b_l fzy singletons of T there exists such that $a_s b_l \subseteq$ Ĥ Suggest that a choice of $a_s \subseteq$ Ĥ or $b_l \subseteq$ Ĥ, $\forall s, l \in L$.

Definition 2.11[2]: Let Ĥ be a fzy ideal of T that is not empty. Then, for any fzy singletons, Ĥ is called the Two-Absorbing Fzy Ideal $a_s,\ b_l,\ r_k$ of T so that $a_s b_l\, r_k \subseteq$ Ĥ suggest that a choice of $a_s b_l \subseteq$ Ĥ or $a_s\, r_k \subseteq$ Ĥ or $b_l\, r_k \subseteq$ Ĥ.

Definition 2.12[2]:Let H be a proper fzy ideal of T. Then H is said to be Two-Absorbing primary fzy ideal of T if $a_s b_l r_h \subseteq H$, implies that $a_s b_l \subseteq H$ or $a_s r_h \subseteq \sqrt{H}$ or $b_l r_h \subseteq \sqrt{H}$ for any fzy singletons $a_s,\ b_l,\ r_h$ of T.

Definition 2.13[4]:A fzy module. Y of a T-module W is named a multiplication fzy module If there is a fzy ideal Ĥ of T such that $P = \hat{H}Y$ for every non-empty fzy sub-module P of Y.

Definition 2.14[3]: Let $P \neq 0_1$ be the fzy second sub-module, and let Y be the fzy module of a T-module W if $\forall\ r \in T$ we have $1_r.P = P$ or $1_r.P = 0_1$ where 1_r is fzy ideal of T.

Definition 2.15 [12]: A fzy module Y of a T-module W is named a comultiplication fzy module if $\mathrm{P} = \mathrm{F} - ann_Y F - \mathrm{ann}P$ for each fzy sub-module P of Y.
Theorem 2.16 [12]: Assume that Ĥ is a non-empty proper fzy ideal of a ring T. Then the following expressions are equivalent:
1- Ĥ is Two-Absorbing fzy ideal of T;
2- If $\underset{\wedge}{U}KD \subseteq$ Ĥ for fzy ideals $\underset{\wedge}{U}, K, D$ of T, $\underset{\wedge}{U}K \subseteq$ Ĥ or $KD \subseteq$ Ĥ or $\underset{\wedge}{U}D \subseteq$ Ĥ.

Definition 1. 17[13]: Let Y be a T-module's fzy. W. A suitable submodule It is claimed that A of Y is an entirely irreducible fzy sub-module if $A =$, where $\{A_i\}_{i\in I}$ is a family of fzy sub-modules of Y, suggest that $A = A_i$ for any $i \in I$. Every fzy sub-module of Y is an intersection of a fully irreducible fzy sub-module of Y, as is readily apparent.

Theorem 2.18[12]:Let $f: X \to$ Y be F-epimorphism of T-module

1) If P is a fzy classical Two-Absorbing sub-module of Y, then $f^{-1}(P)$ is a fzy classical Two-Absorbing sub-module of X.
2) If P is a fzy classical Two-Absorbing sub-module of X containing $ker(f)$. Then $f(P)$ is a fzy classical Two-Absorbing sub-module of Y.

Proposition 2.19 [13]:Let F: X→ Y be Fzy-monomorphism of $\mathrm{T-module}$. If H is a $\mathrm{completely\ irreducible\ fzy\ sub-module}$ of fuzzy module X of a $\mathrm{T-module}$ M, then F(H)is a $\mathrm{completely\ irreducible\ fzy\ sub-module}$ of F(X) .

3. FZY CLASSICAL TWO-ABSORBING SECOND SUB-MODULES.

In this section we introduce the notion of fzy classical Two-Absorbing second sub-modules as a generalization of fzy Two-Absorbing second sub-modules. And studied some definition , examples , theorems and propositions .

Definition 3.1: Let P be a non-zero fzy sub-module of fzy module Y of a T-module W. Then P is named a fzy classical Two-Absorbing second sub-module of Y if whenever a_s, b_l, c_i are fzy singletons of T , H is a completely irreducible fzy sub-module . of Y, and $a_s b_l c_i P \subseteq H$, then $a_s b_l P \subseteq H$ or $b_l c_i P \subseteq H$ or $a_s c_i P \subseteq H$. We say Y is a classical Two-Absorbing fzy second module. if Y is a fzy classical Two-Absorbing second sub-module of itself .

Proposition 3.2: Let $P \neq 0_i$ be fzy sub-module of fzy module Y of a T-module W. Then P is a fzy classical Two-Absorbing second sub-module of Y Iff the level sub-module , $P_u \neq 0$ is a classical Two-Absorbing second sub-module of Y_u for all $u \in L$.

Proof: ⇒) Let $abc\ P_u \subseteq H_u$ for every $a, b, c \in T$ and $P_u \neq 0$ be sub-module of Y_u, H_u be completely irreducible ssub-module of Y_u we have $abc\ y \in H_u$ for all $y \in P_u$, then $H(abcy) \geq u.$ So $(abcy)_u \subseteq H$, implies that $a_s b_k c_i y_l \subseteq H, \forall y_l \in P$ where $u = min\{s, k, i , l\}$, hence $a_s b_k c_i\ P \subseteq H$. Since P is a fzy classical Two-Absorbing second sub-module then either $a_s b_k\ P \subseteq H$ or $b_k c_i\ P \subseteq H$ or $a_s c_i P \subseteq H.$ Hence $a_s b_k y_l \subseteq H$ or $b_k c_i y_l \subseteq H$ or $a_s c_i \subseteq H$, so that $(aby)_u \subseteq H$ or $(bcy)_u \subseteq H$ or $(ac)_u \subseteq H$ .Thus, either $aby \in H_u$ or $bcy \in H_u$ or $ac \in H_u$, $\forall y \in P_u$ so $abP_u \subseteq H_u$ or $bc\ P_u \subseteq H_u$ or $ac \subseteq H_u$ therefore P_u is a classical Two-Absorbing second sub-module of Y_u.

⇐) Let $a_s b_k c_i\ P \subseteq H$ for all fzy singletons a_s , b_k , c_i of T and H is completely irreducible fzy sub-module of Y . Subsequently $a_s\, b_k c_i\, y_l \subseteq H, \forall y_l \in P$,so $(abcy)_u \subseteq H$ where $u = min\{s, k, i , l\}$, hence $H(abcy) \geq u$,then a $bc\ y \in H_u, \forall y \in P_u$ indicates $abc\ P_u \subseteq H_u$, but P_u is a classical Two- Absorbing second sub-module of Y_u ,so that either $abP_u \subseteq H_u$ or $bcP_u \subseteq H_u$ or $acP_u \subseteq H_u$. Subsequently $aby \in H_u$ or $bcy \in H_u$or $acy \in H$, $\forall y \in P_u$ hence either $(aby)_u \subseteq H$ or $(bcy)_u \subseteq H$ or $(ac)_u \subseteq H$ so either $a_s b_k\ P \subseteq H$ or $b_k c_i\ P \subseteq H$ or $a_s c_i P \subseteq H$.

Thus, P is a fzy classical Two-Absorbing second sub-module of Y.

Theorem 3.3: Let P be a non-zero fzy sub-module of Y and Y be the fzy module of a T-module W. Then, the following claims are interchangeable:

a) P is a fzy classical Two-Absorbing second sub-module of Y.
b) For every a_s, b_l fzy singletons of T and completely irreducible fzy sub-module H of Y with $a_s b_l P \nsubseteq H$, $(H: a_s b_l P) = (H: a_s P) \cup (H: b_l P)$.
c) For every a_s, b_l fzy singletons of T and completely irreducible fzy sub-module H of Y with $a_s b_l P \nsubseteq H$, $(H: a_s b_l P) = (H: a_s P)$ or $(H: a_s b_l P) = (H: b_l P)$.
d) For every a_s, b_l fzy singletons of T, every fzy ideal K of T, and completely irreducible fzy sub-module H of Y with $a_s b_l KP \subseteq H$, either $a_s b_l P \subseteq H$ or $a_s KP \subseteq H$ or $b_l KP \subseteq H$.
e) For every a_s fzy singleton of T, every fzy ideal K of T, and completely irreducible fzy sub-module H of Y with $a_s KP \nsubseteq H$, $(H: a_s KP) = (H: KP)$ or $(H: a_s KP) = (H: a_s P)$.
f) For every a_s fzy singleton of T , fzy ideals K, N of T, and completely irreducible fzy sub-module H of Y with $a_s KNP \subseteq H$, either $a_s KP \subseteq H$ or $a_s NP \subseteq H$ or $KNP \subseteq H$.
g) For fzy ideals K, N of T and completely irreducible fzy sub-module H of Y with $KNP \nsubseteq H$, $(H: KNP) = (H: KP)$ or $(H: KNP) = (H: NP)$.
h) For fzy ideals K_1, K_2, K_3 of T, and completely irreducible fzy sub-module H of Y with $K_1 K_2 K_3 P \subseteq H$, either $K_1 K_2 P \subseteq H$ or $K_1 K_3 P \subseteq H$ or $K_2 K_3 P \subseteq H$.
i) For each completely irreducible fzy sub-module H of Y with $P \nsubseteq H$, $(H: P)$ is Two-Absorbing fzy ideal of T.

Proof: $(a) \Rightarrow (b)$ Let $t_v \in (H: a_s b_l P)$. Then $t_v a_s b_l P \subseteq H$. Since $a_s b_l P \nsubseteq H$, then $a_s t_v P \subseteq H$ or $b_l t_v P \subseteq H$ so that $t_v \in (H: a_s P)$ or $t_v \in (H: b_l P)$ hence $(H: a_s b_l P) = (H: a_s P) \cup (H: b_l P)$.

$(b) \Rightarrow (c)$ This follows from the fact that if a fzy ideal is the union of two fzy ideals, then it is equal to one of them.

$(c) \Rightarrow (d)$ Let a_s , b_l be fzy singletons of R and H be a completely irreducible fzy sub-module of Y, such that $a_s b_l KP \subseteq H$. Then $K \subseteq (H: a_s b_l P)$. If $a_s b_l P \subseteq H$, then we are done. Assume that $a_s b_l P \nsubseteq H$. Then by part (c), $K \subseteq (H: b_l P)$ or $K \subseteq (H: a_s P)$ as desired.
$(d) \Rightarrow (e) \Rightarrow (f) \Rightarrow (g) \Rightarrow (h)$ The proofs are similar to those previous implications
$(h) \Rightarrow (i)$ Let K_1 , K_2 , K_3 be fzy ideals of R , $K_1 K_2 K_3 \subseteq (H: P)$, then $K_1 K_2 K_3 P \subseteq H$. By (h) either $K_1 K_2 P \subseteq H$ or $K_1 K_3 P \subseteq H$ or $K_2 K_3 P \subseteq H$, hence either $K_1 K_2 \subseteq (H: P)$ or $K_1 K_3 \subseteq (H: P)$ or $K_2 K_3 \subseteq (H: P)$. So that $(H: P)$ is Two-Absorbing fzy ideal by theorem (2.15)
$(h) \Rightarrow (a)$ Trivial.

Corollary 3.4: Let A be a fzy classical Two-Absorbing second sub-module of a cocyclic T-module W . Then Fzy-ann(P) is Two-Absorbing fzy ideal of T.

Proof: This follows from Theorem (3.3) $(a) \Rightarrow (i)$, because (0) is a completely irreducible fzy sub-module of Y.

Example 3. 5: Let $Y: Z_{p^\infty} \to L$ where $Y(y) = \begin{cases} 1, & y \in Z_{p^\infty} \\ 0, & \text{o.w.} \end{cases}$
in which p can be any prime number. The Y fzy module of Z-module Z_{p^∞} is clearly visible.

Let $S: Z_{p^\infty} \to L$ where $S(y) = \begin{cases} u, & y \in \left(\frac{1}{p^i} + Z\right), i \in I \\ 0, & \text{o.w.} \end{cases}$
S is clearly a fzy sub-module of Y.
Now $S_u = \langle \frac{1}{p^4} + Z \rangle$ is sub-module of $Y_u = Z_{p^\infty}$ as Z-module
S_u is not Two-Absorbing second sub-module Since
$P^3 \langle \frac{1}{p^4} + Z \rangle \subseteq \langle \frac{1}{p} + Z \rangle$ but $P \langle \frac{1}{p^4} + Z \rangle \nsubseteq \langle \frac{1}{p} + Z \rangle$ and $P^3 \nsubseteq ann(S_u) = 0$. So that S is not fzy Two-Absorbing second sub-module of Y by Proposition 3.2. Thus, S is not fzy classical Two-Absorbing second sub-module of Y.

Example 3.6: Let $Y: Z_{pq} \oplus Q \to L$ where $Y(y) = \begin{cases} 1 & y \in Z_{pq} \oplus Q \\ 0 & \text{o.w.} \end{cases}$
where the prime numbers are p and q. The Y fzy module of Z-module $Z_{pq} \oplus Q$ is clearly visible.

Let $S: Z_{pq} \oplus Q \to L$ where $A(y) = \begin{cases} u & y \in Z_{pq} \oplus Q \\ 0 & \text{o.w.} \end{cases}$
It is obvious S a fzy sub-module of Y.
Now $S_u = Z_{pq} \oplus Q$ is a classical Two-Absorbing second sub-module, but it is not a strongly Two-Absorbing second sub-module So that S is a fzy classical Two-Absorbing second sub-module by Proposition (3.2).

Proposition 3.7: Let P be a fzy classical Two-Absorbing second sub-module of fzy module Y of a T-module W. Then we have the following:

a) If a_s fzy singleton of T ,then $a_s^n P = a_s^{n+1} P$, for all $n \geq 2$.
b) If H is a completely irreducible fzy sub-module of Y such that $P \nsubseteq H$, then $\sqrt{(H: P)}$ is Two-Absorbing fzy ideal of R.

Proof:

a) It suffices to demonstrate that $a_s^n P = a_s^{n+1} P$. It is evident that $a_s^3 P \subseteq a_s^2 P$. Let H be a completely irreducible fzy sub-module of Y, such that $a_s^3 P \subseteq H$. Since P is a fzy classical Two-Absorbing second sub-module then $a_s^2 P \subseteq H$. This implies that $a_s^2 P \subseteq a_s^3 P$.
b) Assume that a_s, b_l, c_i are fzy singletons of T and $a_s b_l c_i \subseteq \sqrt{(H: \mathrm{P})}$ Then, a positive integer t exists so $a_s^t b_l^t c_i^t P \subseteq H$. By hypotheses, P is a fzy classical Two-Absorbing second sub-module of Y. Thus, $a_s^t b_l^t P \subseteq H$ or $b_l^t c_i^t P \subseteq H$ or $a_s^t c_i^t P \subseteq H$. Therefore, $b_l \subseteq \sqrt{(H: P)}$ or $b_l c_i \subseteq \sqrt{(H: P)}$ or $a_s c_i \subseteq \sqrt{(H: P)}$.

Theorem 3.8: Let P be fzy sub-module of fzy module Y of a T-module W. Then we have the following:

a) If A is a fzy classical Two-Absorbing second sub-module of fzy module Y of a T-module W, then KP is a fzy classical Two-Absorbing second sub-module of Y for fzy ideal K of T with $K \nsubseteq Fzy - ann(P)$.
b) If P is a fzy classical Two-Absorbing sub-module of Y, then $(P:_Y K)$ is a fzy classical Two-Absorbing sub-module sub-module of Y for fzy ideal K of T with $K \nsubseteq (P: Y)$.
c) Let $\phi: Y_1 \to Y_2$ be F-monomorphism of T-module. If B is a fzy classical Two-Absorbing second sub-module of $\phi(Y_1)$, then $\phi^{-1}(B)$ is a fzy classical Two-Absorbing second sub-module of Y.

Proof:

a) Let K be fzy ideal of T with $K \nsubseteq Fzy - ann(P)$ and a_s, b_l, c_i be fzy singletons of T, H be a completely irreducible fzy sub-module of Y, so that $a_s b_l c_i KP \subseteq H$, then $a_s c_i P \subseteq H$ or $c_i\, b_l KP \subseteq H$ or $a_s b_l KP \subseteq H$ by Theorem (3. 3) $(a) \Rightarrow (d)$. If $c_i\, b_l KP \subseteq H$ or $a_s b_l KP \subseteq H$, then we are done. If $a_s c_i P \subseteq H$, then $a_s c_i KP \subseteq a_s c_i P$ implies that $a_s c_i KP \subseteq H$, as needed. Since $K \nsubseteq \mathrm{Fzy} - \mathrm{ann(P)}$, we have KP is a non-zero fzy sub-module of Y.

b) Use the technique of part (a) and apply Theorem (2.16).

c) If $\phi^{-1}(B) = 0_1$, then $\phi(Y_1) \cap B = \phi\phi^{-1}(B) = \phi\,(0_1) = 0_1$. Thus, $B = 0_1$, is a contradiction. Therefore, $\phi^{-1}(B) \neq 0_1$.Now let a_s, b_l, c_i are fzy singletons of T, H be a completely irreducible fzy sub-module of Y, and $a_s b_l c_i \phi^{-1}(B) \subseteq H$. Then $a_s b_l c_i B = a_s b_l c_i\, (\phi(Y_1) \cap B) = a_s b_l c_i\, \phi\phi^{-1}(B) \subseteq \phi(H)$. By Proposition (2.19), $\phi\,(H)$ is a completely irreducible fzy sub-module of $\phi(Y_1)$. Thus, as B is a fzy classical Two-Absorbing second sub-module $a_s b_l\, B \subseteq \phi(H)$ or $b_l c_i\, B \subseteq \phi(H)$ or $a_s c_i\, B \subseteq \phi(H)$. Therefore $a_s b_l \phi^{-1}(B) \subseteq \phi^{-1}\phi(H) = H$ or $b_l c_i\ \ \phi^{-1}(B) \subseteq \phi^{-1}\phi(H) = H$ or $a_s c_i \phi^{-1}(B) \subseteq \phi^{-1}\phi(H) = H$, as desired .

Corollary 3.9: Let Y be fzy module of a T-module W. Then we have the following:

a) If Y is a fzy multiplication classical Two-Absorbing second T-module then every non-zero fzy sub-module of Y is a fzy classical Two-Absorbing second sub-module of Y.

b) If Y is a comultiplication fzy module and the zero fzy sub-module of Y is a fzy classical Two-Absorbing sub-module, then every proper fzy sub-module of Y is a fzy classical Two-Absorbing sub-module of Y.

Proof: This follows from parts (a) and (b) of Theorem (3.8).

Proposition 3.10: Let Y be fzy module of a T-module W and $\{K_i\}_{i\in I}$ be a chain of fzy classical Two-Absorbing second sub-modules. of Y. Then $\sum_{i\in I} K_i$ is a fzy classical Two-Absorbing second sub-module of Y.

Proof: Let a_s, b_l, c_i are fzy singletons of T, H be a completely irreducible fzy sub-module of Y, and $a_s b_l c_i\ \sum_{i\in I} K_i \subseteq H$. Assume that $a_s b_l \sum_{i\in I} K_i \nsubseteq H$ and $a_s c_i\ \sum_{i\in I} K_i \nsubseteq H$. Then there are $m, n \in I$ that previous implications. where $a_s b_l K_n \nsubseteq H$ and $a_s\ c_i\, K_m \nsubseteq H$. Hence,for every $K_n \subseteq K_s$ and every $K_m \subseteq K_d$ we have that $a_s b_l K_s \nsubseteq H$ and $a_s\ c_i\, K_d \nsubseteq H$. Therefore, for each fzy sub-module K_h such that $K_n \subseteq K_h$ and $K_m \subseteq K_h$, we have $b_l c_i\, K_h \subseteq H$. Hence $b_l\ c_i\ \sum_{i\in I} K_i \subseteq H$, as needed.

Definition 3.11: A fzy classical Two-Absorbing second sub-module P of fzy module Y of a T-module W is a fzy maximal classical Two-Absorbing second sub-module of fzy sub-module B of Y, if $P \subseteq B$ and there does not exist a fzy classical Two-Absorbing second sub-module S of Y such that $P \subset S \subset B$.

Lemma 3.12: Let Y be a T-module W's fzy module. A fzy maximum classical Two-Absorbing second sub-module of Y then contains each of the fzy classical Two-Absorbing second sub-modules of Y.

Proof: Using Zorn's Lemma [6] and Proposition (3.10), this is readily demonstrated.

4. FZY STRONGLY CLASSICAL TWO-ABSORBING SECOND SUB-MODULES.

This section provides a definition of the term fzy strongly classical Two-Absorbing second sub-module of fzy module , we consider the relationship between fzy classical Two-Absorbing second sub-modules and fzy strongly classical Two-Absorbing second sub-modules, example, theorem and propositions

Definition 4.1: Let P be a non-zero fzy sub-module of fzy module Y of a T-module W. We say that P is a fzy strongly classical Two-Absorbing second sub-module of Y if whenever a_s, b_l, c_i are fzy singletons of T, H_1, H_2, H_3 are completely irreducible.fzy sub-modules. of Y and $a_s b_l c_i P \subseteq H_1 \cap H_2 \cap H_3$, then $a_s b_l P \subseteq H_1 \cap H_2 \cap H_3$ or $b_l c_i P \subseteq H_1 \cap H_2 \cap H_3$ or $a_s c_i P \subseteq H_1 \cap H_2 \cap H_3$. We say Y is fzy strongly classical Two-Absorbing second module if Y is a fzy strongly classical Two-Absorbing second sub-module of itself

Note that:

Clearly every fzy strongly classical Two-Absorbing second sub-module is a fzy classical Two-Absorbing second sub-module

Example 4.2: Let $Y\colon Z \to L$ where $Y(\mathrm{y}) = \begin{cases} 1 & \text{if } \mathrm{y} \in \mathrm{Z} \\ \\ 0 & \mathrm{o.w.} \end{cases}$

As Z-module, it is clear that Y is Z's fzy module.

Let $P\colon Z \to L$ where $\mathrm{P(y)} = \begin{cases} \mathrm{u} & \mathrm{y} \in 2\mathrm{Z} \\ 0 & \mathrm{o.w.} \end{cases}$

It evident P is fzy sub-module of Y.

Now, $\mathrm{P} = 2Z$ is not fzy Two-Absorbing second sub-module of $Y_u = Z$ as Z-module, since $2.2.2Z \subseteq 8Z$ where $8Z$ is a completely irred. sub-module of $Y_u = Z$ as Z-module, but $2.2Z \nsubseteq 8Z$ and $2.2 \notin ann(2Z) = (0)$. Therefore, A is not the second sub-module of T-ABSO. Thus, P has no fzy strongly classical Two-Absorbing second sub-module

Theorem 4.3: Let Y fzy module of a T-module W and P be a non-zero fzy sub-module of Y. Then the statements that follow are interchangeable:

a) P is fzy strongly classical Two-Absorbing second sub-module
b) If a_s, b_l, c_i are fzy singletons of T, N is fzy sub-module of Y, and $a_s b_l c_i P \subseteq N$, then $a_s b_l P \subseteq N$ or $b_l c_i P \subseteq N$ or $a_s c_i P \subseteq N$,
c) For every a_s, b_l, c_i are fzy singletons of T, $a_s b_l c_i P = a_s b_l P$ or $a_s b_l c_i P = a_s c_i P$ or $a_s b_l c_i P = b_l c_i P$,
d) For every a_s, b_l are fzy singletons of T and sub-module N of Y with $a_s b_l P \nsubseteq N$, $(N: a_s b_l P) = (N: a_s P) \cup (N: b_l P)$,
e) For every a_s, b_l are fzy singletons of T and fzy sub-module N of Y with $a_s b_l P \nsubseteq N$, $(N: a_s b_l P) = (N: a_s P)$ or $(\mathrm{N}: a_s b_l P) = (N: b_l P)$,
f) For every a_s, b_l are fzy singletons of T, every fzy ideal K of T, and sub-module N of Y with $a_s b_l KP \subseteq N$, either$a_s b_l P \subseteq N$ or $a_s KP \subseteq N$ or $b_l KP \subseteq N$,
g) For every a_s is fzy singleton of T, every fzy ideal K of T, and fzy sub-module N of Y with $a_s KP \nsubseteq N$, $(N: a_s KP) = (N: KP)$ or $(\mathrm{N}: a_s KP) = (N: a_s P)$,
h) For every a_s is fzy singleton of T, fzy ideals K, J of T , and fzy sub-module N of Y with $a_s KJP \subseteq N$, either $a_s KP \subseteq N$ or $a_s JP \subseteq N$ or $KJP \subseteq N$,
i) For fzy ideals K, J of T, and fzy sub-module N of Y with $KJP \nsubseteq N$, $(N: KJP) = (N: KP)$ or $(N: KJP) = (N: JP)$,
j) For fzy ideals K_1, K_2, K_3 of T, and fzy sub-module N of Y with $K_1 K_2 K_3 P \subseteq N$, either $K_1 K_2 P \subseteq N$ or $K_1 K_3 P \subseteq N$ or $K_2 K_3 P \subseteq N$,
k) For each fzy sub-module N of Y with $P \nsubseteq N$, $(N: P)$ is T-ABSO fzy ideal of T.

Proof: $(a) \Rightarrow (b)$ Let a_s, b_l, c_i are fzy singletons of T, N is fzy sub-module of Y, and $a_s b_l c_i P \subseteq N$. Assume on the contrary that $a_s b_l A \nsubseteq N$, $b_l c_i P \nsubseteq N$, and $a_s c_i P \nsubseteq N$. Then there exist completely irreducible fzy sub-modules. H_1, H_2, H_3 of Y such that N is fzy sub-module of them but $a_s b_l P \nsubseteq H_1$, $b_l c_i P \nsubseteq H_2$,and $a_s c_i P \nsubseteq H_3$. Now we have $a_s b_l c_i P \subseteq H_1 \cap H_2 \cap H_3$. Thus, by part (a) , $a_s b_l A \subseteq H_1 \cap H_2 \cap H_3$ or $b_l c_i A \subseteq H_1 \cap H_2 \cap H_3$ or $a_s c_i A \subseteq H_1 \cap H_2 \cap H_3$. Therefore, $a_s b_l A \subseteq H_1$ or $b_l c_i P \subseteq H_2$ or $a_s c_i P \subseteq H_3$ which are contradictions.
$(b) \Rightarrow (c)$ Let a_s, b_l, c_i are fzy singletons of T. Then $a_s b_l c_i P \subseteq a_s b_l c_i P$ implies that $a_s b_l P \subseteq a_s b_l c_i P$ or $b_l c_i P \subseteq a_s b_l c_i P$ or $a_s c_i P \subseteq a_s b_l c_i P$ by part (b). Thus, a $b_l P = a_s b_l c_i P$ or $b_l c_i P = a_s b_l c_i P$ or $a_s c_i P = a_s b_l c_i P$ because the reverse inclusions are clear.
$(c) \Rightarrow (d)$ Let $t_r \in (\mathrm{N}: a_s b_l P)$. Then $t_r a_s b_l P \subseteq N$. Since $a_s b_l P \nsubseteq N$, $a_s t_r P \subseteq N$ or $b_l t P \subseteq N$ as needed.
$(d) \Rightarrow (e)$ This follows from the fact that if fzy ideal is the union of two fzy ideals, then it is equal to one of them.
$(e) \Rightarrow (f)$ Let for some a_s, b_l are fzy singletons of T, fzy ideal K of T , and fzy sub-module N of Y, $a_s b_l KP \subseteq N$. Then $K \subseteq (N: a_s b_l P)$. If $a_s b_l P \subseteq N$, then we are done. Assume that $a_s b_l P \nsubseteq N$. Then by part (d), $K \subseteq (N: b_l P)$ or $K \subseteq (N: a_s P)$ as desired.
$(g) \Rightarrow (h) \Rightarrow (i) \Rightarrow (h) \Rightarrow (j)$ Have proofs similar to that of the previous implications.
$(j) \Rightarrow (a)$ Trivial.
$(j) \Rightarrow (k)$ This is forthright.
Let P be fzy sub-module of fzy module Y of a T-module W. Then Theorem (4.3)
$(a) \Longleftrightarrow (c)$ shows that P is a fzy strongly classical Two-Absorbing second sub-module of Y iff P is a fzy strongly classical Two-Absorbing second module

Corollary 4.4: Let A be a fzy strongly classical Two-Absorbing second sub-module of fzy module Y of a T-module W and K be fzy ideal of T. Then $K^{\mathrm{n}}\mathrm{P} = \mathrm{K}^{\mathrm{n}+1}\mathrm{P}$, for all $n \geq 2$.
Proof: It is enough to show that $K^2\mathrm{P} = \mathrm{K}^3\mathrm{P}$, by Theorem (4.3) , $K^2\mathrm{P} = \mathrm{K}^3\mathrm{P}$.

Proposition 4.5: Let Y fzy module of a T-module W. Then we have the following:

a) If Y is a comultiplication fzy module then P is a fzy strongly classical Two-Absorbing second sub-module of Y.
b) If P_1, P_2 are a fzy quasi-prime second sub-module. of Y, then $P_1 + P_2$ is a fzy strongly classical Two-Absorbing second sub-module of Y.
c) If P is a fzy strongly classical Two-Absorbing second sub-module of Y, then KP is a fzy strongly classical Two-Absorbing second sub-module of Y for all fzy ideals K of T with $K \nsubseteq Fzy - ann(P)$.
d) If Y is a multiplication fzy strongly classical Two-Absorbing second module then every non-zero fzy sub-module of Y is a fzy classical Two-Absorbing second sub-module of Y.
e) If Y is a fzy strongly classical Two-Absorbing second module then every non-zero homomorphic image of Y is a fzy classical Two-Absorbing second module

Proof:

a) By Theorem (4.3) $(a) \Rightarrow (k)$, $Fzy - ann(P))$ is Two-Absorbing fzy ideal of T. Now the result follows from [13, Theorem (4.7)].

b) Let P_1, P_2 be a fzy quasi-prime second sub-module. of Y and a_s, b_l, c_i are fzy singletons of T since P_1 is a fzy quasi-prime second sub-module we may assume that $a_s b_l c_i P_1 = a_s P_1$. Likewise, assume that $a_s b_l c_i P_2 = b_l P_2$. Hence, $a_s b_l c_i (P_1 + P_2) = a_s b_l (P_1 + P_2)$ which implies $P_1 + P_2$ is a fzy strongly classical Two-Absorbing second sub-module by Theorem (4.3) $(c) \Rightarrow (a)$.

c) Use the technique of the proof of Theorem (3.10) (a).

d) This follows from part (c).

e) This is forthright.

Proposition 4.6: Let Y fzy module of a T-module M and $\{K_i\}_{i\in I}$ be a chain of fzy strongly classical Two-Absorbing second sub-modules. of Y. Then $\sum_{i\in I} K_i$ is a fzy strongly classical Two-Absorbing second sub-module of Y.

proof: Use the technique of Proposition (3.10).

Definition 4.7: A fzy strongly classical Two-Absorbing second sub-module P of fzy module Y of a T-module W is a fzy maximal strongly classical Two-Absorbing second sub-module of fzy sub-module N of Y ,if $P \subseteq N$ and there does not exist fzy strongly classical Two-Absorbing second sub-module V of Y such that $P \subset V \subset N$.

Lemma 4.8: Let Y be a T-module W's fzy module. Subsequently, each fzy maximally strongly classical Two-Absorbing second sub-module of Y contains every fzy strongly classical Two-Absorbing second sub-module of Y.

Proof: Proposition (4.6) and Zorn's Lemma [6] make this easily demonstrable.

Theorem 4.9: Let $\phi: Y_1 \to Y_2$ be Fzy-monomorphism of fzy modules Y of an T-modules. Next up is the following:

a) If P is a fzy strongly classical Two-Absorbing second sub-module of Y_1, then $\phi(P)$ is a fzy strongly classical Two-Absorbing second sub-module of Y_2.

b) If B is a fzy strongly classical Two-Absorbing second sub-module of $\phi(Y_1)$, then $\phi^{-1}(B)$ is a fzy strongly classical Two-Absorbing second sub-module of Y_1.

Proof:

a) Since A non-zero fzy sub-module and ϕ is F- monomorphism, we have $\phi(P) \neq 0_1$. Let a_s, b_l, c_i are fzy singletons of T. Then by of Theorem (4.3) $(a) \Rightarrow (c)$, we can assume that $a_s b_l c_i P = a_s b_l P$. Thus, $a_s b_l c_i \phi(P) = \phi(a_s b_l c_i P) = \phi(a_s b_l P) = a_s b_l \phi(P)$. Hence, $\phi(P)$ is a fzy strongly classical Two-Absorbing second sub-module of Y by Theorem (4.3).

b) If $\phi^{-1}(B) = 0_1$, then $\phi(Y_1) \cap B = \phi\, \phi^{-1}(B) = \phi(0_1) = 0_1$. Thus, $B = 0_1$, a contradiction Therefore, $\phi^{-1}(B) \neq 0_1$. Now, let a_s, b_l, c_i are fzy singletons of T, N be fzy sub-module of Y_1 and $a_s b_l c_i \phi^{-1}(B) \subseteq N$. Then $a_s b_l c_i B = a_s b_l c_i (\phi(Y_1) \cap B) = a_s b_l c_i \phi\phi^{-1}(B) \subseteq \phi(N)$. Thus, as B is a fzy strongly classical Two-Absorbing second sub-module $a_s b_l B \subseteq \phi(N)$ or $b_l c_i B \subseteq \phi(N)$ or $a_s c_i B \subseteq \phi(N)$. Therefore, $a_s b_l \phi^{-1}(B) \subseteq \phi^{-1}\phi(N) = N$ or $b_l c_i \phi^{-1}(B) \subseteq \phi^{-1}\phi(N) = N$ or $a_s c_i \phi^{-1}(B) \subseteq \phi^{-1}\phi(N) = N$, as desired.

5. CLASSICAL T-ABSO FUZZY SECONDARY SUB-MODULES

In this section we introduce the concept of fzy classical Two-Absorbing secondary sub-modules. as a dual notion of Two-Absorbing primary fzy sub-modules and getting some related results.

Definition 5.1: A non-zero fzy sub-module P of fzy module Y of a T-module W is a fzy classical Two-Absorbing secondary sub-module of Y, if whenever a_s, b_l are fzy singletons of T, N is fzy sub-module of Y and $a_s b_l P \subseteq N$, then $a_s P \subseteq N$ or $b_l P \subseteq N$ or $a_s b_l \subseteq \sqrt{Fzy - ann(P)}$.

Example 5.2: Every fzy strongly Two-Absorbing second sub-module is a fzy classical Two-Absorbing secondary sub-module but the converser not true in general, for example,

Let $Y: \mathrm{Z}_{\mathrm{p}^\infty} \to \mathrm{L}$ where $Y(y) = \begin{cases} 1 & \mathrm{y} \in \mathrm{Z}_{\mathrm{p}^\infty} \\ 0 & \mathrm{o.w.} \end{cases}$

Where p is any prime integer. It is evident Y fzy module of Z-module $\mathrm{Z}_{\mathrm{p}^\infty}$

Let $A: Z_{p^\infty} \to L$ where $A(y) = \begin{cases} \mathrm{u} & \mathrm{y} \in \ (\frac{1}{\mathrm{p}^3} + \mathrm{Z}) \\ 0 & \mathrm{o.w.} \end{cases}$

It evident is P fzy sub-module of Y.

Now, $\mathrm{A_u} = \langle \frac{1}{\mathrm{p}^3} + \mathrm{Z} \rangle$ is sub-module of $\mathrm{Y_u} = \mathrm{Z}_{\mathrm{p}^\infty}$ as Z-module, $\mathrm{A_u}$ is not Two-Absorbing second sub-module since $\mathrm{P}^2 \langle \frac{1}{\mathrm{p}^3} + \mathrm{Z} \rangle \subseteq \langle \frac{1}{\mathrm{p}} + \mathrm{Z} \rangle$ but $p \langle \frac{1}{\mathrm{p}^3} + \mathrm{Z} \rangle \nsubseteq \langle \frac{1}{\mathrm{p}} + \mathrm{Z} \rangle$ and $\mathrm{P}^2 \nsubseteq ann\left(\langle \frac{1}{\mathrm{p}^3} + \mathrm{Z} \rangle \right) = (0)$

So that, A is not Two-Absorbing second sub-module of Y by [13, Proposition (3.6)]

A is a fzy classical Two-Absorbing secondary sub-module, which is not fzy Two-Absorbing second sub-module of Y.

Example 5.3: Every fzy secondary sub-module. is a fzy classical Two-Absorbing secondary sub-module but the converser not true in general. For example,

Let $Y: A \oplus K \rightarrow L$ where $Y(y) = \begin{cases} 1 & y \in A \oplus K \\ 0 & \text{o.w.} \end{cases}$

The Y fzy module of $A \oplus K$ is clearly the Z-module.

Let $B: A \oplus K \rightarrow L$ where $B(y) = \begin{cases} u & y \in \langle \frac{1}{p} + Z \rangle \oplus \langle \frac{1}{q^2} + Z \rangle \\ 0 & \text{o.w.} \end{cases}$

p and q are prime numbers. Clearly, B is a fzy sub-module of Y.

Now $B_u = \langle \frac{1}{p} + Z \rangle \oplus \langle \frac{1}{q^2} + Z \rangle$ is a classical T-ABSO secondary sub-module of the Z-module $Z_{p^\infty} \oplus Z_{q^\infty}$ but $A \oplus K$ is not secondary sub-module of the Z-module $Z_{p^\infty} \oplus Z_{q^\infty}$, Then $A \oplus K$ is not fzy secondary sub-module of the Z-module $Z_{p^\infty} \oplus Z_{q^\infty}$, but $A \oplus K$ is a fzy classical Two-Absorbing secondary sub-module of the Z-module $Z_{p^\infty} \oplus Z_{q^\infty}$

Theorem 5.4: Let P be a non-zero fzy sub-module of fzy module Y of a T-module W. The statements that follow are interchangeable:

a) P is a fzy classical Two-Absorbing secondary sub-module of Y.

b) If $KJP \subseteq N$ for some fzy ideals K, J of T and fzy sub-module N of Y, then $KP \subseteq N$ or $JP \subseteq N$ or $KJ \subseteq \sqrt{Fzy - ann(P)}$

c) For each a_s, b_l are fzy singletons of T, we have $a_s b_l \mathrm{P} = a_s P$ or $a_s b_l \mathrm{P} = b_l P$ or $a_s b_l \subseteq \sqrt{Fzy - ann(p)}$.

Proof: $(a) \Rightarrow (b)$ Let P be a a fzy classical Two-Absorbing secondary sub-module of Y and let $KJP \subseteq N$ for some fzy ideals K, J of T and fzy sub-module N of Y. Suppose $KJ \nsubseteq \sqrt{Fzy - ann(p)}$.. Then for some $a_s \subseteq K$ and $b_l \subseteq J$, $a_s b_l \nsubseteq \sqrt{Fzy - ann(P)}$. Now since $a_s b_l P \subseteq N$, $a_s P \subseteq N$ or $b_l P \subseteq N$. We show that either $KP \subseteq N$ or $JP \subseteq N$. On contrary, We assume that $KP \nsubseteq N$ and $JP \nsubseteq N$. Then there exist $a_{s1} \subseteq K$ and $b_{l1} \subseteq J$ such that $a_{s1} P \nsubseteq N$ and $b_{l1} P \nsubseteq N$. Since $a_{s1} b_{l1} P \subseteq N$ and P is a fzy classical Two-Absorbing secondary sub-module then $a_{s1} b_{l1} \subseteq \sqrt{F - ann(A)}$ We have the following three cases:

Case 1: Suppose $a_s P \subseteq N$ but $b_l P \nsubseteq N$. Since $a_{s1} b_l P \subseteq N$ and $b_l P \nsubseteq N$ and $a_{s1} P \nsubseteq N$, we have $a_{s1} b_l \subseteq \sqrt{Fzy - ann(P)}$.. Now, $(a_s + a_{s1}) b_l P \subseteq N$ and $a_s P \subseteq N$ but $a_{s1} P \nsubseteq N$ therefore $(a_s + a_{s1}) P \nsubseteq N$. As$(a_s + a_{s1}) b_l P \subseteq N$ and $b_l P \nsubseteq N$, $(a_s + a_{s1}) P \nsubseteq N$. implies $(a_s + a_{s1}) b_l \subseteq \sqrt{Fzy - ann(P)}$.Thus, $a_{s1} b_l \subseteq \sqrt{Fzy - ann(P)}$. implies that $a_s b_l \subseteq \sqrt{Fzy - ann(P)}$, a contradiction.

Case 2: Suppose $b_l P \subseteq N$ but $a_s P \nsubseteq N$. Then similar to the Case 1, we get a contradiction.

Case 3: Suppose $a_s P \subseteq N$ and $b_l P \subseteq N$. Now, $b_l P \subseteq N$ and $b_{l1} P \nsubseteq N$ imply $(b_l + b_{l1}) P \nsubseteq N$. Since $a_s (b_l + b_{l1}) P \subseteq N$ and $(b_l + b_{l1}) P \nsubseteq N$ and $a_{s1} P \nsubseteq N$, we get $a_{s1} (b_l + b_{l1}) \subseteq \sqrt{Fzy - ann(P)}$.Since $a_{s1} b_{l1} \subseteq \sqrt{Fzy - ann(P)}$. we have $a_{s1} b_l \subseteq \sqrt{Fzy - ann(P)}$. Again, $P \subseteq N$ and $a_{s1} P \nsubseteq N$ imply $(a_s + a_{s1}) P \nsubseteq N$. Since $(a_s + a_{s1}) b_l P \subseteq N$ and $(a_s + a_{s1}) P \nsubseteq N$ and $b_{l1} P \nsubseteq N$, we have $(a_s + a_{s1}) b_l \subseteq \sqrt{Fzy - ann(P)}$.. Now, as $a_{s1} b_{l1} \subseteq \sqrt{Fzy - ann(P)}$. we get $a_s b_{l1} \subseteq \sqrt{Fzy - ann(P)}$. Since $(a_s + a_{s1})(b_l + b_{l1}) P \subseteq N$ and $(a_s + a_{s1}) P \nsubseteq N$ and $(b_l + b_{l1}) P \nsubseteq N$, we have $(a_s + a_{s1})(b_l + b_{l1}) \subseteq \sqrt{Fzy - ann(P)}$. Since $a_s b_{l1}$, $a_{s1} b$, $a_{s1} b_{l1} \subseteq \sqrt{Fzy - ann(P)}$., we have $a_s b_l \subseteq \sqrt{Fzy - ann(P)}$., a contradiction. Hence, $KP \subseteq N$ or $JP \subseteq N$

$(b) \Rightarrow (c)$ Let a_s, b_l are fzy singletons of T. Then $a_s b_l P \subseteq a_s b_l P$ implies that $a_s P \subseteq a_s b_l P$ or $b_l P \subseteq a_s b_l P$ or $a_s b_l \subseteq \sqrt{Fzy - ann(P)}$. Thus, $a_s b_l P = a_s P$ or $a_s b_l P = b_l P$ or $a_s b_l \subseteq \sqrt{Fzy - ann(P)}$.

$(c) \Rightarrow (a)$ This is clear.

Remark 5.5: Let P and N are two fzy sub-modules. of fzy module Y of a T-module W. To prove $P \subseteq N$, it is enough to show that if H is a completely irreducible fzy sub-module of Y such that $N \subseteq H$, then $P \subseteq H$.

Theorem 5.6: Let A be a fzy classical Two-Absorbing secondary sub-module of fzy module Y of a T-module W. Then Fzy-ann(P) is Two-Absorbing primary fzy ideal of T.

Proof: Let a_s, b_l, c_i are fzy singletons of T and $a_s b_l c_i \subseteq \mathrm{Fzy - ann(P)}$

Suppose that $a_s b_l \nsubseteq \mathrm{Fzy - ann(P)}$ and $b_l c_i \nsubseteq \sqrt{\mathrm{Fzy - ann(P)}}$ we show that $a_s c_i \subseteq \sqrt{\mathrm{Fzy - ann(P)}}$. There exist completely irreducible fzy sub-modules. H_1 and H_2 of Y so that $a_s b_l P \nsubseteq H_1$ and $b_l c_i P \nsubseteq H_2$. Since $a_s b_l c_i P = 0_1 \subseteq H_1 \cap H_2$, $a_s c_i P \subseteq (H_1 \cap H_2 :_Y b_l)$. Thus, $b_l\, a_s P \subseteq H_1 \cap H_2$ or $c_i\, b_l P \subseteq H_1 \cap H_2$ or $a_s c_i \subseteq \sqrt{\mathrm{Fzy - ann}(P)}$. If $b_l a_s P \subseteq H_1 \cap H_2$ or $c_i\, b_l P \subseteq H_1 \cap H_2$, then $b_l\, a_s P \subseteq H_1$ or $c_i\, b_l P \subseteq H_2$ which are contradictions. Therefore, $a_s c_i \subseteq \sqrt{\mathrm{Fzy - ann(P)}}$.

Proposition 5.7: If K is Two-Absorbing primary fzy ideal ideal of T, then $\sqrt{K}$ is Two-Absorbing fzy ideal of T [11]

Corollary 5.8: Let *A* be a fzy classical Two-Absorbing secondary sub-module of fzy module *Y* of a T-module *W*. Then $\sqrt{Fzy - ann(P)}$ is Two-Absorbing fzy ideal of *T*.

Proof: By Theorem (5.6), $Fzy - ann(P)$ is Two-Absorbing primary fzy ideal of *T*. Thus, by Proposition (5.7), $\sqrt{Fzy - ann(P)}$ is Two-Absorbing fzy ideal of *T*.

Example 5.9: The converse of Theorem (5.6) is not true in general for example.

Let $Y: Z_{pq} \oplus Q \to L$ where $Y(y) = \begin{cases} 1 & y \in Z_{pq} \oplus Q \\ 0 & \text{o.w.} \end{cases}$

It is evident *Y* is fzy module of $Z_{pq} \oplus Q$ as Z-module

Let $A: Z_{pq} \oplus Q \to L$ where $A(y) = \begin{cases} u & y \in Z_{pq} \oplus Q \\ 0 & \text{o.w.} \end{cases}$

p and q are prime numbers. Clearly, A is a fzy sub-module of Y.

Now $ann(y) = 0$ is Two-Absorbing Primary ideal of Z , But *Y* is not a fzy classical Two-Absorbing secondary Z-module Then $Fzy - ann(y) = 0_1$ is Two-Absorbing primary fzy ideal of *Z*, But *Y* is not a fzy classical Two-Absorbing secondary Z-module

Theorem 5.10: Let *P* be fzy sub-module of fzy module *Y* of a T-module *W*. Then we have the following:

a) If *A* is a fzy classical Two-Absorbing secondary sub-module of *Y* then *KA* is a fzy classical Two-Absorbing secondary sub-module of *Y* for all ideals *K* of *T* with $K \nsubseteq Fzy - ann(P)$.

b) If *Y* is a multiplication fzy classical Two-Absorbing secondary module Then every non-zero fzy sub-module of *Y* is a fzy classical Two-Absorbing secondary sub-module of *Y*.

Proof:

a) Let *K* be fzy ideal of *T* with $K \nsubseteq Fzy - ann(P)$. Then *KP* is a non-zero fzy sub-module of *Y*. *Let* a_s, b_l are fzy singletons of *T*, *N* be fzy sub-module of *Y* and $a_s b_l KP \subseteq N$. then $a_s b_l P \subseteq (N :_Y \text{K})$. Thus, $a_s KP \subseteq N$ or $b_l KP \subseteq N$ or $a_s b_l \subseteq \sqrt{Fzy - ann(P)} \subseteq \sqrt{Fzy - ann(KP)}$. As needed.

b) This follows from part (a).

Theorem 5.11: Let : $Y_1 \to Y_2$ be Fzy-monomorphism of T-module. Then we have the following:

a) If *P* is a fzy classical Two-Absorbing secondary sub-module of Y_1, then $\phi(P)$ is a fzy classical Two-Absorbing secondary sub-module of Y_2.

b) If *B* is a classical T-ABSO fzy secondary sub-module of $\phi(Y_1)$, then $\phi^{-1}(B)$ is a fzy classical Two-Absorbing secondary sub-module of Y_1.

Proof:

a) Since *P* is non-zero fzy sub-module and *f* is Fzy-monomorphism, we have $\phi(P) \neq 0_1$. Let a_s, b_l are fzy singletons of *T*, *B* be fzy sub-module of *Y*, and $a_s b_l\, \phi(P) \subseteq B$. Then $a_s b_l\, P \subseteq \phi^{-1}(B)$. As *P* is fzy classical Two-Absorbing secondary sub-module $a_s P \subseteq \phi^{-1}(B)$ or $b_l\, P \subseteq \phi^{-1}(B)$ *or* $a_s b_l \subseteq \sqrt{Fzy - ann(P)}$. Therefore, $a_s \phi(P) \subseteq \phi\big(\phi^{-1}(B)\big) = \phi(Y_1) \cap B \subseteq B$ or $b_l\, \phi(P) \subseteq \phi\big(\phi^{-1}(B)\big) = \phi(Y_1) \cap B \subseteq B$ or $a_s b_l \subseteq \sqrt{Fzy - ann(f(P))}$, As needed.

b) If $\phi^{-1}(B) = 0_1$, then $\phi(Y_1) \cap \text{B} = \phi\phi^{-1}\text{B} \acute{=} \phi(0_1) = 0_1$. Thus, $\text{B} = 0_1$, a contradiction . Therefore, $\phi^{-1}(B) \neq 0_1$. Now let a_s, b_l are fzy singletons of *T*, *N* be fzy sub-module of *X* and $a_s b_l\, \phi^{-1}(\text{B}) \subseteq N$. Then $a_s b_l\, \acute{B} = a_s b_l\, (\phi(Y_1) \cap \text{B}) = a_s b_l\, \phi\phi^{-1}(\text{B}) \subseteq \phi(N)$. As B is fzy classical Two-Absorbing secondary sub-module $a_s B \subseteq \phi(N)$ or $b_l \text{B} \subseteq \phi(N)$ or $a_s b_l \sqrt{Fzy - ann(B)}$. Hence, $a_s \phi^{-1}(B) \subseteq \phi^{-1}\, \phi(N) = N$ or $b_l \phi^{-1}(\text{B}) \subseteq \phi^{-1}\phi(N) = N$ or $a_s b_l \subseteq \sqrt{Fzy - ann\big(\phi^{-1}(B)\big)}$, As desired.

Conflicts Of Interest

The author's paper declares that there are no relationships or affiliations that could create conflicts of interest.

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