



Research Article

Fuzzy Semi and Fuzzy Strongly Semi Two-Absorbing Second Sub-modules

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ABSTRACT

In this work, fuzzy semi Two-Absorbing second sub-modules and fuzzy strongly semi Two-Absorbing second sub-modules are presented. The connections between fuzzy semi prime (fuzzy strongly semi prime) second sub-modules, fuzzy quasi prime (fuzzy strongly quasi prime) second sub-modules, and fuzzy Two-Absorbing (fuzzy strongly Two-Absorbing) second sub-modules are also covered. Here are some fundamental characteristics and attributes of these ideas.

1. INTRODUCTION

Zadeh [16] first put forward fuzzy sets in 1965. Fuzzy groups, a generalization of the prime fuzzy ideal, were first proposed by Rosenfeld [1] in 1971. In 1989, Mukherjee [14] extended this notion to the prime fuzzy ideal. In 1992, Zahedi [17] introduced the idea of a main fuzzy ideal. In 2004, Hadi [4] expanded this idea to include semi-primitive fuzzy sub-modules. The prime fuzzy sub-module notion was first presented by Rabi [14]. The definition of the quasi-prime fuzzy sub-module was first put forth by Hatam in [5] in 2001. In 2017, Deniz S. et al. [2] introduced the idea of the Two-Absorbing fuzzy ideal.

In 2018 and 2019, Hatam and Wafaa introduced the notions of two-absorbing fuzzy sub-modules [6] and semi-two-absorbing fuzzy sub-modules [7]. In the year 2019, H. Ansari Toroghy [3] proposed fuzzy second sub-modules. Wafaa and Assel originally introduced the idea of fuzzy two-absorbing second sub-modules in 2021 [15].

There are three parts to the search. The definition and investigation of the Two-Absorbing fuzzy second sub-module are presented in Section (1). In section (2), along with several hypotheses, theorems, and examples, we define fuzzy semi Two-Absorbing second sub-modules and list the necessary features. The fuzzy strongly semi-two-absorbing second sub-module is examined in section (3). Its properties and relationships with other fuzzy second sub-module concepts are searched for in the remarks and examples.

Note that: The notations fzy sub-module, fzy singleton, and fzy module represent the fuzzy sub-module, fuzzy singleton, and fuzzy module.

2. PRELIMINARIES

Definition 1.1[16]: Let S be a non-empty set and L be an interval $[0,1]$ of the real line (real number). A function from S into L is a fzy set A in S (a fzy subset of S).

Definition 1.2[8]: Let $x_u: S \rightarrow L$ be a fzy set in S , where $x \in S, u \in L$, define by

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$$x_u(y) = \begin{cases} u & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}, x_u \text{ is known as fzy singleton in } S, \text{ If } x = 0 \text{ and } u = 1, \text{ then } 0_1(y) = \begin{cases} 1 & \text{if } y = 0 \\ 0 & \text{if } y \neq 0 \end{cases}$$

Definition 1.1.8[18]: A fzy ideal of a ring T is a fzy subset K of that ring, if for all $x, y \in T$:

- 1- $K(x - y) \geq \min\{K(x), K(y)\}$.
- 2- $K(xy) \geq \max\{K(x), K(y)\}$.

Definition 1.3[18]: Let W be a T -module fzy set Y of W is called fzy module of a T -module W if

- 1- $Y(x - y) \geq \min\{Y(x), Y(y)\}$, for all $x, y \in W$.
- 2- $Y(rx) \geq Y(x)$, for all $x \in W, r \in T$.
- 3- $X(0) = 1$ (0 is the zero element of W).

Definition 1.4[9]: Let Y and A be two fzy modules of a T -module W . A is called fzy sub-module of Y if $A \subseteq Y$.

Proposition 1.5[12]: Let A be fzy set of W . Then the level subset $A_u, \forall u \in L$ is a sub-module of M if and only if A is fzy submodule of fzy module of a T -module W .

Definition 1.6[18]: Let A and B be two fzy sub-modules of fzy module Y . The residual quotient of A and B denoted by $(A : B)$ is the fzy subset of T defined by:

$(A : B)(r) = \sup\{v \in L : r_v \cdot B \subseteq A\}$ for all $r \in T$. That $(A : B) = \{r_v : r_v \cdot B \subseteq A ; r_v \text{ is a fzy singleton of } T\}$. If $B = \langle x_k \rangle$, then $(A : \langle x_k \rangle) = \{r_v : r_v \cdot x_k \subseteq A ; r_v \text{ is fzy singleton of } T\}$.

Definition 1.7 [9]: Assume that A is a proper fzy sub-module of Y . The definition of the fzy annihilator of A , represented by $F\text{-ann}A$, is: $(F - \text{ann}A)(r) = \sup\{v : v \in L, r_v A \subseteq 0_1\}$, To all $r \in T$.

Note that: $F - \text{ann}A = (0_1 :_R A)$, hence $(F - \text{ann}Y)_v \subseteq \text{ann}Y_v$, [5].

Proposition 1.8 [9]: $F\text{-ann}Y$ is the fzy ideal of T if Y is the fzy module of a T -module W .

Definition 1.9 [11]: A ring T 's prime fzy ideal is its fzy ideal \hat{H} , provided that H is non-empty and that for any a_s, b_l fzy singleton of T , $a_s b_l \subseteq \hat{H}$ suggests that a choice of $a_s \subseteq \hat{H}$ or $b_l \subseteq \hat{H}, \forall s, l \in L$.

Definition 1.10 [2]: Let \hat{H} be a fzy ideal of R that is not empty. Therefore, \hat{H} is referred to as the Two-Absorbing fzy ideal if, for any fzy singletons a_s, b_l, r_k of T then $a_s b_l r_k \subseteq \hat{H}$ Suggest that a choice of $a_s b_l \subseteq \hat{H}$ or $a_s r_k \subseteq \hat{H}$ or $b_l r_k \subseteq \hat{H}$.

Definition 1.11 [8]: If B is any fzy sub-module of X that contains A , then $B = X$. A maximum fzy sub-module is a correct fzy sub-module A of a fzy module X of a T -module W .

Definition 1.12[3]: Let $A \neq 0_1$ be the fzy second sub-module of a T -module W , and let Y be its fzy module $\forall r \in R$ we've $1_r \cdot A = A$ or $1_r \cdot A = 0_1$ where 1_r is fzy ideal of T .

Definition 1.13[13]: A divisible fuzzy module is a fzy module Y of a T -module W if each fzy singleton $0_1 \neq r_v$ of R such that $r_v \cdot Y = Y$.

Definition 1. 14[15]: Let Y be a T -module's fzy. W . A suitable submodule It is claimed that A of Y is an entirely irreducible fzy sub-module if $A = \bigcap_{i \in I} A_i$, where $\{A_i\}_{i \in I}$ is a family of fzy sub-modules of Y , suggest that $A = A_i$ for any $i \in I$. Every fzy sub-module of Y is an intersection of a fully irreducible fzy sub-module of Y , as is readily apparent.

Definition 1.15[15]: If fzy singleton a_s of T and B are entirely irreducible fzy sub-modules, then let $A \neq 0_1$ be designated as fzy prime (fzy Strongly prime) second sub-module (B be fzy sub-module). $a_s A \subseteq B$, then $A \subseteq B$ or $a_s \subseteq F - \text{ann}(A)$.

Definition 1.16[15]: Let $A \neq 0_1$ be fzy sub-module of fzy module of Y of a T -module W . A is named fzy Two-Absorbing second sub-module if each time fzy singletons a_s, b_l of T , B is completely irreducible fzy sub-module and $a_s b_l A \subseteq B$ then either $a_s A \subseteq B$ or $b_l A \subseteq B$ or $a_s b_l \subseteq F - \text{ann}(A)$.

Definition 1.17[15]: Let $A \neq 0_1$ be fzy sub-module of fzy module of Y of a T -module W . A is named fzy quasi-prime (fzy strongly quasi-prime) second sub-module if each time fzy singletons a_s, b_l of R , B is completely irreducible fzy sub-module (B is fzy sub-module) and $a_s b_l A \subseteq B$ then either $a_s A \subseteq B$ or $b_l A \subseteq B$.

Remarks 1.18[15]: The fzy Two-Absorbing second sub-module is clearly the same as every fzy quasi-prime second sub-module.

3. FZY SEMI TWO-ABSORBING SECOND SUB-MODULES

In this part, the fzy semi-two-absorbing second sub-module notion will be examined along with its properties, theorems, assertions, notes, and examples.

Definition 2.1: A non-zero fzy sub-module A of fzy module Y of a T -module W is known as fzy semi prime second sub-module if whenever $a_s^2 A \subseteq K$ where a_s is fzy singleton of T , $s \in L$ and K is completely irreducible fzy sub-module of Y , implies $a_s A \subseteq K$.

Proposition 2.2: Let $A \neq 0_1$ be a fzy sub-module of fzy module Y of a T -module W . Then A is a fzy semi prime second sub-module of Y if and only if the level A_u is semi prime second sub-module of Y_u , for all $u \in L$.

Proof: \Rightarrow) Let $a^2 A_u \subseteq K_u$ for every $a \in R$ and $A_u \neq 0$ be a submodule of Y_u, K_u be a completely irreducible sub-module of Y_u , we have $aA_u \subseteq K_u$ then $K(aA) \geq u$. So $(a^2 A)_u \subseteq K$ implies that $a_s^2 A_u \subseteq K$. Since A is a fzy semi prime second sub-module, then $a_s A \subseteq K$ equival $a_s^2 A \subseteq K$. Hence, $(aA)_u \subseteq K$ and $(a^2 A)_u \subseteq K$, so that $aA_u \subseteq K_u$ and $a^2 A_u \subseteq K_u$. Thus, A_u is a semi prime second sub-module.

\Leftarrow) Let $a_s^2 A \subseteq K$ for fzy singletons a_s of $T, \forall s \in L$, hence $(a^2 A)_u \subseteq K$ so that $K(a^2 A) \geq u$ implies $a^2 A \subseteq K_u$, but A_u is a semi prime second sub-module, then $aA_u \subseteq K_u$ and $a^2 A_u \subseteq K_u$. Hence, $(aA)_u \subseteq K$ and $(a^2 A)_u \subseteq K$ implies $a_s A \subseteq K$ and $a_s^2 A \subseteq K$. Thus, A is fzy semi prime second sub-module.

Example 2.3: Let $Y: Z_{10} \rightarrow L$ where $Y(y) = \begin{cases} 1 & \text{if } y \in Z_{10} \\ 0 & \text{o. w.} \end{cases}$

It is clear Y is a fzy module of Z_{10} as Z -module

Now, $Y_u = Z_{10}$ as Z -module is semi prime second sub-module since $5^2 Y_u \subseteq (\bar{5})$, then $5^2 Y_u = 5Y_u = (\bar{5})$. So that, Y is fzy semi prime second sub-module.

Definition 2.4: A non-zero fzy sub-module A of fzy module Y of a T -module W , is called fzy semi Two-Absorbing second sub-module of Y if whenever $a_s^2 A \subseteq K$ where a_s is fzy singleton of R , $s \in L$ and K is a completely irreducible fzy sub-module, implies either $a_s A \subseteq K$ or $a_s^2 \subseteq F - ann(A)$. (es either bmdubmd.) of Y

The following proposition specificities of fzy semi Two-Absorbing second sub-module is given.

Proposition 2.5: Let $A \neq 0_1$ be a fzy sub-module of fzy module Y of a T -module W . Then A is a fzy semi Two-Absorbing second sub-module of Y if and only if the level A_u is a semi Two-Absorbing second sub-module of Y_u , for all $u \in L$.

Proof: \Rightarrow) Let $a^2 A_u \subseteq K_u$ for every $a \in R$ and $A_u \neq 0$ be sub-module of Y_u, K_u be a completely irreducible sub-module of $Y_u, \forall u \in L$, then $K(a^2 A) \geq u$, hence $(a^2 A)_u \subseteq K$ so that, $a_s^2 A_v \subseteq K$ where $u = \min\{s, v\}$ and $(a^2)_s = a_s^2$ but A is a fzy semi Two-Absorbing second sub-module, then either $a_s A \subseteq K$ or $a_s^2 \subseteq F - ann(A)$. Hence, $(aA)_u \subseteq K_u$ or $(a^2)_u \subseteq ann(A_u)$, implies $aA_u \subseteq K_u$ or $a^2 \in ann(A_u)$. Thus, A_u is a semi Two-Absorbing second sub-module of Y_u .

\Leftarrow) Let $a_s^2 A \subseteq K, a_s$ is a fzy singleton of T and K be completely irreducible fzy sub-module of Y , then $(a^2 A)_u \subseteq K$ where $u = \min\{s, 1\}$, hence $K(a^2 A) \geq u$ so that, $a^2 A_u \subseteq K_u$. But A_u is semi Two-Absorbing second sub-module of Y_u , then either $aA_u \subseteq K_u$ or $a^2 \in ann(A_u)$, hence $(aA)_u \subseteq K$ or $(a^2)_u \subseteq ann(A_u)$, so that $a_s A \subseteq K$ or $a_s^2 \subseteq F - ann(A)$. Thus, A is a fzy semi Two-Absorbing second sub-module of Y .

Remarks and Examples 2.6:

1- In general, the opposite is not true; for instance, every fzy-semi prime second sub-module is fzy-semi Two-Absorbing second sub-module:

$$\text{Let } Y: Z_9 \rightarrow L \text{ where } Y(y) = \begin{cases} 1 & \text{if } y \in Z_9 \\ 0 & \text{o. w.} \end{cases}$$

It is clear Y is a fzy module of Z_9 as Z -module

Now, $Y_u = Z_9$ as Z -module is a semi Two-Absorbing second sub-module since $3^2 Y_u = 0$ but it is not semi prime second sub-module since $3^2 Y_u = (0)$ but $3Y_u \neq (0)$ so that, Y is fzy semi Two-Absorbing second sub-module, but it is not fzy-semi prime second sub-module

2- Every fzy Two-Absorbing second sub-module is fzy semi Two-Absorbing second sub-module. However, In general, the opposite is untrue. For instance:

$$\text{Let } Y: Z_{20} \rightarrow L \text{ where } Y(y) = \begin{cases} 1 & \text{if } y \in Z_{20} \\ 0 & \text{o. w.} \end{cases}$$

As a Z -module, it is clear that Y is a fzy module of Z_{20} .

Now, $Y_u = Z_{20}$ as Z -module, for all $u \in L$, is semi Two-Absorbing second sub-module since $5^2 (Y_u) \subseteq (\bar{5})$, then $5(Y_u) \subseteq (\bar{5})$, but it is not Two-Absorbing second sub-module since $2.5(Y_u) \subseteq (10)$, but $2.(Y_u) \not\subseteq (10)$, $5.(Y_u) \not\subseteq (10)$ and $2.5 \notin ann(Y_u) = 20Z$. So that, Y is a fzy- semi Two-Absorbing second sub-module, which it is not fzy Two-Absorbing second sub-module of Y .

3- Every fzy quasi prime second sub-module is a fzy semi Two-Absorbing second sub-module, However, as demonstrated by the example in part (2), the opposite is generally false.

Proposition 2.7: Let $A \neq 0_1$ be a fzy sub-module of fzy module Y of a T -module W . Then the following expressions are equivalent:

- 1- A is fzy semi Two-Absorbing second sub-module and $F-ann(A)$ is a fzy semi prime ideal.
- 2- A is a fzy prime second sub-module.

- 3- A is a fzy semi prime second sub-module.
- 4- A is a fzy quasi prime second sub-module.
- 5- A is fzy Two-Absorbing second sub-module and $F\text{-ann}(A)$ is a fzy prime ideal.

Proof: (1) \rightarrow (2)

Let $a_s(a_sA) \subseteq K$ for fzy singleton a_s of R . Since A is fzy semi Two-Absorbing second sub-module, then $a_sA \subseteq K$ or $a_s^2 \subseteq F - \text{ann}(A)$. If $a_sA \subseteq K$ then we are done. If $a_s^2 \subseteq F - \text{ann}(A)$ then $a_s \subseteq F - \text{ann}(A)$ since $F - \text{ann}(A)$ is a fzy semi prime ideal. So that, A is a fzy prime second sub-module.

(1) \rightarrow (3)

Let $a_s^2A \subseteq K$ for fzy singleton a_s of T . Since A is fzy Two-Absorbing second sub-module, then $a_sA \subseteq K$ or $a_s^2 \subseteq F - \text{ann}(A)$. If $a_sA \subseteq K$ the proof is complete. If $a_s^2 \subseteq F - \text{ann}(A)$, then $a_s \subseteq F - \text{ann}(A)$ since $F\text{-ann}(A)$ is a fzy semi prime ideal. Hence, $a_sA \subseteq K$. Thus, A is a fzy semi prime second sub-module.

(2) \rightarrow (3)

Let $a_s^2A \subseteq K$ for fzy singleton a_s of T . Since A is a fzy prime second sub-module, then $a_sA \subseteq K$ then we are done.

(3) \rightarrow (4) Let $a_s(a_sA) \subseteq K$ for fzy singleton a_s of T . Since A is a fzy semi prime second submodule, then $a_sA \subseteq K$, shows that A is a second submodule of fzy quasi prime.

(4) \rightarrow (5) Since A is a fzy quasi-prime second submodule, then A is a fzy Two-Absorbing second sub-module by Remarks (1.18)

Now, let $a_s b_l \subseteq F - \text{ann}(A)$, for fzy singletons a_s, b_l of R then $a_s b_l A \subseteq 0_1$, hence $a_s A \subseteq 0_1$ or $b_l A \subseteq 0_1$. Thus, $a_s \subseteq F - \text{ann}(A)$ or $b_l \subseteq F - \text{ann}(A)$, so that $F\text{-ann}(A)$ is a fzy prime ideal.

(5) \rightarrow (1) It is clear.

4. FZY STRONGLY SEMI TWO-ABSORBING SECOND SUBMODULE

In this section, the definition of the strongly semi Two-Absorbing fzy second sub-module is discussed, along with some of the associated conclusions and proofs.

Definition 3.1: A non-zero fzy submodule A of fzy module Y of a T -module W , is called a fzy strongly semi Two-Absorbing second submodule if whenever $a_s^2 A \subseteq K$ never led strongly semi T-ABSO F.second subm, where a_s fzy singleton of T and K fzy sub-module of Y , implies either $a_s A \subseteq K$ or $a_s^2 \subseteq F - \text{ann}A$ Equivalently if $A \neq 0_1$ fzy sub-module is called a fzy strongly semi Two-Absorbing second submodule $a_s^2 A = a_s A$ or $a_s^2 A = 0_1$ for all a_s fzy singleton of R .

Proposition 3.2: Let $A \neq 0_1$ be a fzy sub-module of fzy module Y of a T -module. Then A is a fzy strongly semi Two-Absorbing second sub-module of Y if and only if the level A_u is a strongly semi Two-Absorbing second sub-module of Y_u , for all $u \in L$.

Proof: The same proof of Proposition (2.5) only replaces K is a completely irreducible fzy sub-module by K is a fzy submodule of Y .

Proposition 3.3 : Let Y be a fzy module of a T -module M . A non-zero fzy sub-module A of Y is a fzy strongly semi Two-Absorbing second submodule of Y if and only if $(K :_R A)$ is a fzy semi prime ideal of T for each fzy sub-module $A \not\subseteq K$ in Y and $a_s^2 A \neq 0_1$ for each fzy singleton a_s of T .

Proof:

\Rightarrow) Let $A \neq 0_1$ be a fzy strongly semi Two-Absorbing second submodule of Y and K fzy sub-module Y such that $A \not\subseteq K$ implies $(K : A)$ be a proper fzy ideal of T . Let a_s be fzy singleton of T such that $a_s^2 \in (K : A)$, then $a_s^2 A \subseteq K$ and $a_s^2 A \neq 0_1$, then $a_s A \subseteq K$. Hence, $a_s \in (K : A)$. Thus, $(K : A)$ is a fzy semi prime ideal of T .

\Leftarrow) Let K be a fzy sub-module of Y and $(K : A)$ be fzy semi prime, let a_s be fzy singleton of T such that $a_s^2 A \subseteq K$ ingleton of R such that. In case $A \subseteq K$, then $a_s A \subseteq K$. If $A \not\subseteq K$ and $a_s^2 \in (K : A)$, then $a_s \in (K : A)$ since $(K : A)$ is a fzy semi prime ideal, so that $a_s A \subseteq K$. Thus, A is a fzy strongly semi Two-Absorbing second sub-module.

Corollary 3.4 : If A is a fzy strongly semi Two-Absorbing second sub-module, then $F\text{-ann}(A)$ is a fzy semi prime ideal of T .

Proof: By Proposition (3.3), it is complete proof.

Remark 3.5: In general, the opposite of Corollary (3.4) is not true, as demonstrated by:

Let $Y: Z \rightarrow L$ where $Y(y) = \begin{cases} 1 & \text{if } y \in Z \\ 0 & \text{o. w.} \end{cases}$

Since Z is a Z -module, it is clear that Y is a fzy module of Z .

Let $A: Z \rightarrow L$ where $A(y) = \begin{cases} u & \text{if } y \in PZ \\ 0 & \text{o. w.} \end{cases}$

P is a prime number in this case. A 's status as a fzy submodule of Y is clear.

Now, $A_u = PZ$ and $Y_u = Z$ as Z -module, $ann(A_u) = 0$ is a semi prime ideal of Z , but A_u is not strongly semi Two-Absorbing second sub-module since $P^2A_u \subseteq P^2A_u$ but $PA_u \not\subseteq P^2A_u$ and $P^2 \notin ann(A_u) = 0$. So that $F - ann(A) = 0_1$ is a semi prime fzy ideal, but A is not fzy strongly semi Two-Absorbing second sub-module of Y .

Proposition 3.6: Anon zero fzy sub-module A of fzy module Y of a T -module W is a fzy strongly semi Two-Absorbing second sub-module if and only if for each fzy ideal H of t and for fzy sub-module K of Y such that $H^2A \subseteq K$ and $H^2A \neq 0_1$ implies $HA \subseteq K$.

Proof:

\Rightarrow) Let A be a fzy strongly semi Two-Absorbing second submodule of Y . Then $A \neq 0_1$, let H be a fzy ideal and K fzy submodule of Y . If $A \not\subseteq K$, then either $H^2A \not\subseteq K$ and so nothing to prove or $H^2A \subseteq K$, hence $H^2 \subseteq (K :_R A)$ and by Proposition (3.3) is fzy semi prim ideal of R , so we have $H \subseteq (K :_R A)$, then $HA \subseteq K$.

\Leftarrow) It is clear.

Remarks and Examples 3.7:

1- Every fzy strongly semi prime second sub-module is fzy strongly semi Two-Absorbing second submodule, However, generally speaking, the opposite is not true. For instance:

$$\text{Let } Y: Z_4 \rightarrow L \text{ where } Y(y) = \begin{cases} 1 & \text{if } y \in Z_4 \\ 0 & \text{o. w.} \end{cases}$$

It is clear Y is a fzy module of Z_4 as Z -module

Now, $Y_u = Z_4$ as Z -module is strongly semi Two-Absorbing second sub-module since $2^2Y_u = 0$ but it is not strongly semi prime second sub-module since $2^2Y_u = (0)$ but $2Y \neq (0)$ so that, Y is a fzy strongly semi Two-Absorbing second sub-module, but it is not strongly semi prim fzy second sub-module.

2- It is evident that the second submodule that is fzy strongly quasi-prime is also fzy strongly semi Two-Absorbing; however, this is not always the case. For instance:

$$\text{Let } Y: Z_6 \rightarrow L \text{ where } Y(y) = \begin{cases} 1 & \text{if } y \in Z_6 \\ 0 & \text{o. w.} \end{cases}$$

It is clear that Y is fzy module of Z_6 as Z -module

$$\text{Let } A: Z_6 \rightarrow L \text{ where } Y(y) = \begin{cases} \frac{1}{2} & \text{if } y \in Z_6 \\ 0 & \text{o. w.} \end{cases}$$

It is clear A is a fzy sub-module of Y .

A is a fzy strongly semi Two-Absorbing second sub-module since $a_s^2A = a_sA$ for each fzy singleton of Z , However, the second sub-module is not a fzy strongly quasi-prime one since $\frac{2}{3}A, \frac{3}{3}A = 0_1 \subseteq 0_1$ but $\frac{2}{3}A \not\subseteq 0_1$ and $\frac{3}{3}A \not\subseteq 0_1$.

3- Every fzy second sub-module is a fzy strongly semi Two-Absorbing second sub-module, but the converse incorrect in general, for example, see the example in part (2) where A is a fzy strongly semi Two-Absorbing second sub-module but it is not fzy second submodule since $a_sA \neq A$ for each a_s fzy singleton of Z .

4- A fzy secondary sub-module is a fzy strongly semi Two-Absorbing second sub-module. The conversely is not true, see example part (2) A is a fzy strongly semi Two-Absorbing second sub-module, but it is not a fzy secondary sub-module since $a_sA \neq A$ and $a_s^2A \neq 0_1$.

5- It is obvious that a fzy strongly Two-Absorbing second sub-module is a fzy strongly semi Two-Absorbing second sub-module. In general, the opposite is not true. For instance:

$$\text{Let } Y: Z_6 \oplus Z_{p^\infty} \rightarrow L \text{ where } Y(y) = \begin{cases} 1 & \text{if } y \in Z_6 \oplus Z_{p^\infty} \\ 0 & \text{o. w.} \end{cases}$$

Where p is a prime number, it is clear that Y is a fzy module of $Z_6 \oplus Z_{p^\infty}$ as Z - module. Now, $Y_u = Z_6 \oplus Z_{p^\infty}$ as Z -module is strongly semi Two-Absorbing second submodule since $a^2Y_u = aY_u$ for each $a \in Z$, but it is not strongly Two-Absorbing second submodule since $2.3Y_u = 0 \oplus Z_{p^\infty}$ we have neither $2Y_u \subseteq 0 \oplus Z_{p^\infty}$, nor $3Y_u \subseteq 0 \oplus Z_{p^\infty}$ and nor $2.3Y_u \subseteq (0)Y_u$. So that, Y is a fzy strongly semi Two-Absorbing second submodule, but it is not fzy strongly Two-Absorbing second sub-module.

6- If A is maximal fzy sub-module and hence (prime fzy sub-module) then A may not be fzy strongly semi Two-Absorbing second sub-module for example:

$$\text{Let } Y: Z \rightarrow L \text{ where } Y(y) = \begin{cases} 1 & \text{if } y \in Z \\ 0 & \text{o. w.} \end{cases}$$

It is clear Y is a fzy module of Z as Z - module.

$$\text{Let } A: Z \rightarrow L \text{ where } Y(y) = \begin{cases} \frac{1}{2} & \text{if } y \in PZ \\ 0 & \text{o. w.} \end{cases}$$

Where p is a prime number, it is clear A is a fzy submodule of Y .

A is a maximal fzy submodule of Y, but A is not strongly semi Two-Absorbing fzy second submodule since $a_s^2 A \neq a_s A$ and $a_s^2 A \neq 0_1$ for each $a_s \neq 0_1$ fzy singleton of Z.

- 7- Let A and B be fzy submodules of fzy module Y of a T-module W, with $A \subseteq B \subseteq Y$. If B is a fzy strongly semi Two-Absorbing second sub-modules then A may not be a fzy strongly semi Two-Absorbing second sub-module of Y, for example:

Let $Y: Q \rightarrow L$ where $Y(y) = \begin{cases} 1 & \text{if } y \in Q \\ 0 & \text{o. w.} \end{cases}$

It is clear that Y is a fzy module of Q as Z-module.

Let $A: Q \rightarrow L$ where $A(y) = \begin{cases} \frac{1}{2} & \text{if } y \in Z \\ 0 & \text{o. w.} \end{cases}$

It is clear that A is a fzy submodule of Y

Let $B = Y$ where $A \subseteq B \subseteq Y$, since B is divisible fzy module if X is a fzy divisible sub-module of itself.

Then B is fzy strongly semi Two-Absorbing second sub-module but the fzy sub-module A is not fzy strongly semi Two-Absorbing second sub-module since $a_s^2 A \neq a_s A$ and $a_s^2 A \neq 0_1$ for each $a_s \neq 0_1$ fzy singleton of Z.

Conflicts Of Interest

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