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Research Article

Investigation of Bounded Modules and Some related Concepts

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ABSTRACT

The definition of a class of module as well as submodule namely (bounded module and bounded submodule) has introduced in various ways. In this study, we concentrate on some comparisons and analyzations of bounded module's properties and its relation specifically with prime module and other types of modules and submodules such as: scalar, multiplication, and cyclic modules. In addition, we scan the behavior of this concept under some conditions in order to reach other types of modules and given some counterexamples for several unsatisfying relationships. Finally, we add some new results about some types of modules that are related to bounded modules and open a new path that encourage other researchers about this topic.

1. INTRODUCTION

In this paper, the ring T is commutative with identity and a T-module Ω is unitary. The concept of bounded module was introduced and defined by some authors using some different tools. In 1976, Carl Faith gave the definition of bounded T-module where (a T-module Ω is said to be bounded if there exists $x \in \Omega$ such that $\mathrm{ann}_{\mathrm{T}}(\mathrm{x}) = \mathrm{ann}_{\mathrm{T}}(\Omega)$ [5]. Also, if $\mathrm{ann}_{\mathrm{T}}(\Omega) = B$, then Ω is called B-bounded.

Along with the context of Carl Faith definition, Buthyna Najad introduced the notion of almost bounded submodule where (a submodule N is said to be almost bounded if there exists an element $x \in \Omega, x \notin N$ such that $\mathrm{ann}_{\mathrm{T}}(\mathrm{x}) = \mathrm{ann}_{\mathrm{T}}(\mathrm{N})$[17]. Likewise, Adwia Jassim introduced a generalization of the notion of bounded module called semi-bounded module where (a T-module Ω is said to be semi-bounded T-module if there exists $x \in \Omega$ such that $\sqrt{ann_T(x)} = \sqrt{ann_T(\Omega)}$[18].

Based on [2] the bounded module defined as follow, a T-module Ω has to be bounded if $P^n\Omega$ =0 for some $n \in Z^+$. In 2016, (Pat Goeters Overtoun M.G.Jenda) defined a bounded module using an essential ideal so a T-module Ω is called bounded provided $R/ann_T(\Omega)$ is a right bounded ring where if each essential right ideal of a ring T contains $0 \neq P\ of\ T\ such\ that\ P$ is an essential right ideal of T then if this satisfy we say that T is right bounded.[3]

Moreover, a module Ω that belongs to the category of left T-module is called bounded if there exist $0 \neq r \in T$ such that $r \in ann_T(\Omega)$[4]. Also, according to Heakyung Lee, a T-module Ω is called bounded if for any $N \leq_e \Omega$ there exists an ideal P of T with $P/ann_T(\Omega) \leq_e T/ann_T(\Omega)$ here P is an right ideal such that ΩP ⊆ N.[4]

Later, Al-Ani, studied in some details the definition of bounded T-module that belongs to Carl Faith and gave some properties including the definition of bounded submodule where (a submodule N of a T-module is called bounded if there exists $x \in N$ such that $ann_T(x) = ann_T(N)$[6]. Further, Al-Ani, said that Ω is called fully bounded T-module if every non-zero submodule of Ω is bounded.[6]. In 1978, Beachy and Blair presented the concept namely (finitely annihilated module) which is a generalization of bounded module for Carl Faith where (a T-module Ω is called finitely annihilated if $ann_T(\Omega)$ is the annihilator of a finite subset.

In other word, when the following relation $ann_T(\Omega) = \mathrm{ann}_{\mathrm{T}}(a_1, a_2, \ldots, a_n), a_i \in \Omega$, $\mathrm{i} = 1,2,3,\ldots,\mathrm{n}$ holds then Ω is called finitely annihilated T-module [23]. Recently, Mijbass and Habat gave an equivalent characterization of finitely annihilated module when a T-module Ω is called finitely annihilated if the following equality holds

$ann_T(\Omega) = \mathrm{ann}_{\mathrm{T}}(A)$ for some finitely generated T-submodule A of Ω [24]. It is obvious that every bounded T-module is finitely annihilated T-module, but the converse is not true in general and later we are going to show that in some details.

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However, in order to connect the bounded module with other types of module like a prime module where (a T-module Ω is called prime if for each proper submodule A of Ω we have $ann_T(\Omega) = ann_T(A)$ or equivalently for every $x, y \in \Omega$ the relation $ann_T(x) = ann_T(y)$ holds.[35]. As a result, algebraically the definition in [1] is much relevant than others definitions that we mentioned earlier since Carl Faith definition deals with the following ideal

$ann(\Omega) = \{ r \in R : rx = 0, \forall x \in \Omega\}$ that play an essential role in order to obtain a prime module in some ways. Carl concept of bounded module and prime module merge somehow in the structure of finitely annihilated T-module since both bounded and prime module are finitely annihilated and this notion is great problem to find more relationship between them. The paper consist of three sections where the first section include the summary of the definitions of bounded module in some different prospective. In the second section, we present and clarify some basic properties and remarks while in the third section giving a close relation for some types of modules and submodules with bounded module. Finally, we conclude with some notes related to bounded module and some suggestions that will be useful for other researcher.

2 BASIC DEFINITIONS

In this section, we give some essential definitions of bounded module in terms of several path that each path has a different properties.

Definition. 2.1 [1] Let Ω be a T-module, then Ω is called bounded T-module whenever $x \in \Omega$, implies

$$ann_T(x) = ann_T(\Omega). \tag{1}$$

The definition of bounded module above belongs to Carl Faith and it is clear that the ideal

$ann_T(\Omega) = \{ r \in T : rx = 0, \forall x \in \Omega \}$ plays a significant role to satisfy the bounded module condition in (1). In addition, the definition providing that there exists an element in Ω as we are going to see in the following example.

Example 2.2 Let $\Omega = Z_2 \oplus Z_2$ as Z-module is bounded module since there exists $x = (\bar{0}, \bar{1}) \in Z_2 \oplus Z_2$ such that $ann_T(x) = ann_T(\Omega) = 2Z$. Moreover, if we let $ann_T(\Omega) = A$, then Ω is called A-bounded. [5]

In the same context of Carl definition of bounded module , Al-Ani present the definition of bounded submodule and studied some properties that we will talk about them in some details later.

Definition 2.3 [2] Let Ω be a T-module and $A < \Omega$ where A is a proper submodule of Ω, then A is said to be bounded if there exists $a \in A$ such that

$$ann_T(a) = ann_T(\mathrm{A}). \tag{2}$$

Furthermore, a T-module Ω is called fully bounded module if Ω is bounded and every non-zero submodule A of Ω is also bounded.[28]

Example 2.4 Suppose that $\Omega = Z_3 \oplus Z$, T=Z and $A = \langle \bar{0} \rangle \oplus 3Z$, then A is bounded submodule since there exists an element $a = (\bar{0}, 3) \in A$ such that $ann_T(a) = ann_T(\mathrm{A}) = \langle \bar{0} \rangle$.

In the equation (2) if $a \in \Omega, a \notin A$, then A is called almost bounded submodule and this concept has been introduced by Buthyna Najad and the definition as follow

Definition 2.5 [17] Let Ω be a T-module and $A < \Omega$ where A is a proper submodule of Ω, then A is said to be almost bounded if there exists $a \notin A, a \in \Omega$ such that

$$ann_T(a) = ann_T(\mathrm{A}). \tag{3}$$

It is obvious from the definition that every bounded submodule is almost bounded and the converse is not true.

Example 2.6 $\Omega = \mathrm{Z}_{12}$ as Z-module and $A = \langle 3 \rangle$, then there exists $(\bar{3}) \in \Omega, (\bar{3}) \notin A$ such that $ann_T(\bar{3}) = ann_T(\mathrm{A}) = 4Z$ so A is almost bounded submodule.

Definition 2.7 [18] Let Ω be a T-module, then Ω is called semi-bounded T-module if there exists $x \in \Omega$, such that

$$\sqrt{ann_T(x)} = \sqrt{ann_T(\Omega)}. \tag{4}$$

A semi-bounded module consider as a generalization of bounded module that introduced by Adwia in 2012 and it is clear that every bounded module is semi-bounded.

Let $\Omega = Z \oplus Z_6$ as Z-module is semi-bounded module since there exists $x = (\bar{1}, \bar{0}) \in Z \oplus Z_6$ such that

$\sqrt{ann_T(x)} = \sqrt{ann_T(\Omega)}$. [3] and Q as Z-module is bounded module so it is also semi-bounded module. As a generalization of bounded module that Carl in [1] mentioned, Blair and Beach established the notion of finitely annihilated module in 1978 where a T-module Ω is said to finitely annihilated if $ann_T(\Omega)$ is equal to the annihilator of a finite subset of Ω.

Definition 2.8 [22] Let Ω be a T-module, then Ω is called finitely annihilated module if there exists $a_1, a_2, \dots \dots .., a_n \in \Omega$ such that

$$ann_T(\Omega) = ann_T(a_1, a_2, \dots \dots .., a_n). \tag{5}$$

Later, Mijbass presented an equivalent definition of finitely annihilated module as follow

Definition 2.9 [24] Let Ω be a T-module, then Ω is called finitely annihilated module if there exists a finitely generated submodule A of Ω such that

$$ann_T(\Omega) = ann_T(\mathrm{A}) \tag{6}$$

Also, from the definition (6) and (1) we see that every bounded module is finitely annihilated module but the converse is not true.

The main idea of our systematic review paper is concentrate about these definitions of bounded module from (1-6) and we see that all different version of bounded module ((1-6) are almost derived from Carl Faith definition which use the annihilator of the module Ω so many properties in common between them as we will see in the next section. However, we will give some other definition of bounded module belong to other authors using different tools briefly and the details leave it the reader.

Definition.2.10 A T-module Ω is said to be bounded if $\frac{T}{ann_T(\Omega)}$ is a right bounded ring where a ring T is right bounded if for every essential right ideal p of T contains a non-zero ideal I of T such that I is essential right ideal of T [2].

Definition 2.11 A T-module Ω that belong to the category of left T-module which denote it by TMod is called bounded if there exist $0 \neq r \in T$ such that $r\Omega = 0$ [26].

Definition 2.12 [4], A T-module Ω is called bounded module if for any $A \leq \Omega$ where A is an essential submodule of Ω there exist an ideal I of T with $\frac{I}{ann(\Omega)}$ essential as right ideal in $\frac{T}{ann(\Omega)}$ such that $MI \subseteq$ A.

Moreover, the definition of bounded module in [27] define Ω as follow

Definition 2.13 A T-module Ω is said to be bounded if every essential submodule of Ω contains a fully invariant essential submodule and a T-module Ω is called fully bounded if $\Omega/$K is bounded for every $K \in Spec(\Omega)$ where the Spec Ω is the set of all prime submodules of Ω.

On other word, Ω is fully bounded if for every prime submodule and any submodule A of Ω with $K \subsetneq A$, there exists a fully invariant essential submodule H of Ω such that $K \subsetneq H \subseteq$ A [27].

Definition 2.14 A T-module Ω is bounded if $ann_T\left(\frac{\Omega}{\mathrm{A}}\right) \leq_e T_T$ for all essential submodule A of Ω and Ω is called fully bounded if ($\Omega/$K) is bounded as module over $R/ann_T(\frac{M}{K})$ for any L_2-prime submodule of Ω[20].

For more information about the definitions (1.10 – 1.14) in details we refer the reader to [1, 2, 3, 4, 21]

The rest of the article will deals specifically with definition of bounded module that belong to Carl Faith [1] with its generalization and we are going to give some properties, examples, and remarks. Also, we will involve the concept of prime submodule with bounded module and explain the relationship between them in the next section.

3 SOME BASIC PROPERTIES AND REMARKS

In this section, we give some essential properties of bounded module and discuss some certain modules that have same characterization or involve properly in bounded module.

Recall Ω is bounded module if there exists $x \in \Omega$ such that $ann_T(x) = ann_T(\Omega)$. [5]

Note that T is a commutative ring with identity and Ω is unitary T-module and $A \leq \Omega$ means that A is a submodule of Ω.

Remark 3.1 The class of bounded module is not closed under taking submodule.

In general, if Ω is bounded module then it is not necessary that every submodule A of Ω is also bounded and for that the next example shows this notion.

Example 3.2 Consider $\Omega = Z_4$ $as\ Z - module$ and let $A = (\bar{2})$. Then Ω is bounded Z-module but A is not bounded submodule.

However, there are some conditions that make a bounded T-module is closed under taking submodule.

Recall the following concepts

- A T-module Ω is said to be multiplication T-module if for each submodule A of Ω there exists an ideal I of T such that $A = I\Omega$ [12]
- A T-module Ω is said to be divisible if and only if $r\Omega = \Omega$, $\forall\ 0 \neq r \in T$. [10]
- A T-module is called faithful module if $ann_T(\Omega) = 0$ [10].
- Let T be an integral domain and Ω be a T-module, then $T(\Omega) = \{x \in \Omega, \exists\ 0 \neq r \in T, rx = 0\}$, $T(\Omega)$ is called torsion submodule if $T(\Omega) = \Omega$ and if $T(\Omega) = 0$, then Ω is called torsion-free [15]
- A T-module Ω is called uniform module if every non-zero submodule of Ω is an essential.[5].

Proposition 3.3 Suppose that a T-module Ω where (T is integral domain) is a torsion-free module. Then every non-zero proper submodule of Ω is bounded.

Proof. Assume that Ω is torsion-free module then by [6], Ω is bounded. Now, let A be any non-zero proper submodule of Ω, then A is also torsion-free submodule.

Therefore, A is bounded.

Corollary 4.3 Suppose that a T-module Ω where (T is integral domain) is

1. Faithful multiplication T-module
2. Projective T-module
3. Faithful cyclic T-module
4. Divisible multiplication T-module
5. Free T-module
6. Z_P where p is prime.

Then Ω is bounded and every non-zero proper submodule of Ω is bounded.

Corollary 3.5 Let Ω be a bounded faithful fully stable T-module, then every submodule of Ω is bounded[5].

Proposition 3.6 Suppose that Ω is a T-module, $0 \neq a \in \Omega$ such that $Ta \leq_e \Omega$ and $ann_T(a)$ is prime ideal of T with $ann_T(a) = ann_T(\Omega)$. Then Ω is fully bounded T-module [5].

Corollary. 3.7 Let Ω be a bounded uniform T-module and $A < \Omega$ such that $ann_T(\Omega)$ is prime ideal of T, then A is bounded T-submodule.[5]

Corollary 3.8 Let Ω be a bounded uniform faithful T-module and $A < \Omega$, then A is bounded T-submodule.

Proposition 3.9 Let Ω, Ω' be two T-modules and let $\varphi: \Omega \to \Omega'$ be an isomorphism. Then

1- If A is bounded submodule of Ω, then $\varphi(N)$ is bounded submodule of Ω'.

2- If D is bounded submodule of Ω', then $\varphi^{-1}(D)$ is bounded submodule of Ω.

Proof. 1) Suppose that A is bounded submodule, then there exists $x \in A \subseteq \Omega$ such that $ann_T(x) = ann_T(A)$.

Thus, $\varphi(x) \in \varphi(A)$ implies that $ann_T\big(\varphi(A)\big) \subseteq ann_T\varphi(x)$. Now, let $r \in ann_T(\varphi(x))$, then $r\varphi(x) = 0$ so $\varphi(rx) = 0$ implies that $rx = 0$ and $r \in ann(x) = ann(A)$. Therefore, rA=0 and hence $\varphi(rA) = 0$, then $r\varphi(A) = 0$. We conclude that $r \in ann_T\big(\varphi(A)\big)$.

2) Assume that D is bounded submodule, then there exists $y \in D$ such that $ann_T(y) = ann_T(D)$. Since φ is epiomorphism then there exists $x \in \Omega$ such that $\varphi(x) = y$. So $x \in \varphi^{-1}(D)$, then $ann_T\big(\varphi^{-1}(D)\big) \subseteq ann_T(x)$.

Let $r \in ann_T(x), then\ rx = 0$ implies that $\varphi(rx) = 0$ and $r\varphi(x) = 0$. Therefore,

$r \in ann_T(\varphi(x)) = ann_T(y) = ann_T(D)$, then rD=0 and implies that $\varphi^{-1}(rD) = 0$ and $r\varphi^{-1}(D) = 0$ which means that $r \in ann_T(\varphi^{-1}(D))$.

Proposition 3.10 Let Ω, Ω' be two bounded T-modules, then $\Omega \oplus \Omega'$ is bounded T-module.

Proof. Since Ω, Ω' are bounded modules, then there exists $x \in \Omega$, such that $ann_T(x) = ann_T(\Omega)$ and $y \in \Omega'$ such that $ann_T(y) = ann_T(\Omega')$.

Thus $(x, y) \in \Omega \oplus \Omega'$ so we have $ann_T(\Omega \oplus \Omega')$. Now, let $r \in ann_T(x, y)$ implies that $(rx, ry) = (0,0)$ so rx=0 and ry=0 . Therefore, $r \in ann_T(x) = ann_T(\Omega)$ and $r \in ann_T(y) = ann_T(\Omega')$. Hence, $r \in ann_T(\Omega) \cap \mathrm{ann}_T(\Omega')$ implies that

$r \in ann_T(\Omega \oplus \Omega')$.

Note that a direct summand of bounded module is not necessary to be bounded module and the next example will show that.

Let $\Omega = Z \oplus Z_{P^\infty}$ as Z-module is bounded module but Z_{P^∞} is not bounded Z-module.

Proposition 3.11 Let Ω_1, Ω_2 be two T-modules and $\Omega = \Omega_1 \oplus \Omega_2$. If A_1 and A_2 are bounded submodule of $\Omega_1 and\ \Omega_2$ respectively, then $A_1 \oplus A_2$ is bounded submodule of Ω.

Proof. Let A_1, A_2 be bounded submodules of Ω_1, Ω_2 respectively, then there exist $x \in A_1$ such that

$ann_T(x) = ann_T(A_1)$ and $y \in A_2$ such that $ann_T(y) = ann_T(A_2)$. Therefore, $(x, y) \in A_1 \oplus A_2$. Thus,

$ann_T(x, y) = ann_T(x) \cap ann_T(y) = ann_T(A_1) \cap ann_T(A_2) = ann_T(A_1 \oplus A_2)$. Hence $A_1 \oplus A_2$ is bounded submodule of Ω.

Corollary 3.12 Let A_1, A_2 be two bounded submodules of T-module Ω. Then $A_1 \oplus A_2$ is bounded submodule of $\Omega \oplus \Omega$.

Corollary 3.13 Let Ω be a T-module and A be a bounded submodule of Ω. Then $A^2 = A \oplus A$ is a bounded submodule of $\Omega^2 = \Omega \oplus \Omega$.

Proposition 3.14 Let Ω be a torsion-free T-module where (T is an integral domain), then Ω is bounded.

Also, a multiplication faithful T-module gives a torsion-free module and hence it is bounded.

Remark 3.15 If Ω/A is bounded T-module, then it is not necessary that Ω be bounded in general.[5]

Let $\Omega = Z_2 \oplus A$ as Z-module where $A = \oplus_{P>2} Z_P$, then Ω is not bounded but $\Omega/A \approx Z_2$ is bounded Z-module.[5]

However , there some conditions that make remark (3.15) true.

Recall a submodule A of a T-module Ω is called pure if $I\Omega \cap A = IA$ for every ideal I of T and if T is principle ideal domain or Ω is a cyclic, then A is pure if and only if $r\Omega \cap A = rA$,

$\forall r \in T$. [16]

Proposition 3.16 Let Ω be a T-module and A be a pure submodule of Ω such that Ω/A is bounded T-module and $[A :_T \Omega] = \mathrm{ann}_T(\Omega)$. Then Ω is bounded.

Proof. See [5].

Corollary 3.17 Let Ω be a F-regular T-module such that Ω/A is bounded T-module and $[A :_T \Omega] = \mathrm{ann}_T(\Omega)$. Then Ω is bounded.

Corollary 3.18 Let A be a submodule of T-module Ω such that every finitely generated submodule of A is pure in Ω. If Ω/A is bounded T-module and $[A :_T \Omega] = \mathrm{ann}_T(\Omega)$., then Ω is bounded.

Corollary 3.19 Let Ω be a T-module where (T is a TID) and A is a divisible submodule of Ω such that Ω/A is bounded T-module and $[A :_T \Omega] = \mathrm{ann}_T(\Omega)$. Then Ω is bounded.

Note that the purity of A is an essential condition and we can think of other assumptions that lead to a pure submodule.

Lemma 3.20 Let Ω/A be a torsion-free T-module where T is PID, then A is pure submodule of Ω.

As result, if Ω is faithful T-module where T is PID and Ω/A is torsion-free, then Ω is bounded.

Proposition 3.21 Every cyclic T-module is bounded.[5]

Corollary 3.22 Every simple T-module is bounded.

Note that the converse of proposition (3.21) and corollary (3.22) is not true in general. If we consider Q as Z-module is bounded but not cyclic and Z as Z-module is also bounded but it is not simple.

Recall a T-module is said to cyclic over End(Ω) if there exist $x \in \Omega$ such that for every element $y \in \Omega$ we have

$\varphi(x) = y$ for some $\varphi \in End(\Omega)$. And every cyclic T-module is also cyclic over its endomorphism.[5]

Remark 3.23 If a T-module is cyclic over its endomorphism then it is bounded. [5]

Proof. Let $x \in \Omega$, then $ann_T(\Omega) \subseteq ann_T(x)$. Now, let $r \in ann_T(x)$. Since Ω is cyclic over End(Ω), then $rx = 0$ implies that $\varphi(rx) = 0, so\ r\varphi(x) = 0$. Hence, $rm = 0$ for each $m \in \Omega$. Therefore, $r \in ann_T(\Omega)$.

Thus, from the remark 2.23 we conclude that the endomorphism of a module plays also in some way in the notion of bounded module and we can rise the following question

Question 3.24 If we have an endomorphism of a T-module Ω $\varphi: \Omega \longrightarrow \Omega$ defined in some way. How the form of bounded module would be?

Question 3.25 If Ω is bounded T-module over the endomorphism of Ω structure. Does every submodule of Ω is bounded as well?

Recall a T-module Ω is called fully stable module if $ann_\Omega\big(ann_T(x)\big) = (x), \forall x \in \Omega.$ [11]

Remark 3.26 Let Ω be a bounded fully stable T-module, then Ω is cyclic over End(Ω).[5]

Remark 3.27 Let Ω be a bounded quasi-injective T-module, then Ω is cyclic over End(Ω).[5]

Next propositions and corollaries show the relationship between bounded module and a class of module called distinguished module.

Recall a T-module Ω is said to distinguished module if $ann_\Omega(I) \neq 0$ for every maximal ideal I of T.[14]

Proposition 3.28 Let Ω be a distinguished T-module such that $ann_T(\Omega)$ is maximal ideal of T, then Ω is bounded.[14]

Proof. Suppose that Ω is a distinguished T-module and $ann_T(\Omega)$ is prime ideal of T, then by [10] there exist $0 \neq x \in \Omega$ such that $ann_T(x) = ann_T(\Omega)$ and this means that Ω is bounded T-module.

Recall A ring T is said to be 2-regular ring if every ideal of T is 2-pure ideal where an ideal I of a ring T is said to be 2-pure ideal if for every ideal P of T we have $P^2 \cap I = P^2 I.$ [33]

Corollary 3.29 Let T be a 2-regular ring and Ω is a distinguished T-module such that $ann_T(\Omega)$ is prime ideal of T. Then Ω is bounded T-module [14].

Proof. Since $ann_T(\Omega)$ is prime ideal and T is 2-regular, then by [33], $ann_T(\Omega)$ is maximal ideal of T and then by proposition 3.23, we get the result.

Corollary 3.30 Let Ω be a distinguished T-module such that $ann_T(\Omega)$ is prime and semimaximal ideal in T. Then Ω is bounded T-module [14].

Proof. Since $ann_T(\Omega)$ is prime and semimaximal ideal in T, then by [32] $ann_T(\Omega)$ is maximal ideal of T. Then by proposition 3.23, the result is follow.

4 BOUNDED MODULES RELATED TO PRIME MODULES

Recall a T-module is called prime module if $ann_T(A) = ann_T(\Omega)$ for every submodule A of Ω [16].

It is clear that every prime module is bounded but the converse is not true in general and to see that let $\Omega = Z_6$ as Z-module is bounded Z-module but it is not prime since $ann_Z(\Omega) = 6Z$ but $ann_Z(\overline{2}) = 3Z$.

However, in order that connect bounded module with prime we need some conditions and the next proposition and corollaries will show that.

Proposition 4.1 Let Ω be a T-module and $0 \neq x \in \Omega$ such that Ω is bounded and $ann_T(x)$ is prime ideal of T with Rx is an essential submodule of Ω. Then Ω is prime T-module [5].

Proof. Let A be a submodule of Ω, then there exists $0 \neq t \in T, 0 \neq tx \in A$. Thus

$ann_T(\Omega) = ann_T(x) \subseteq ann_t(A) \subseteq ann_T(tx)$.

Let $r \in ann_T(tx), then\ r(tx) = 0$. Hence $rx = 0$ and $r \in ann_T(x)$ implies that $r \in ann_T(A)$. Therefore

$ann_T(A) = ann_T(tx) \subseteq ann_T(x) = ann_T(\Omega)$. The proof is now complete.

Corollary 4.2 Let Ω be a uniform bounded T-module such that $ann_T(\Omega)$ is prime ideal of T. Then Ω is prime module[5].
Corollary 4.3 Let Ω be a uniform T-module such that $ann_T(\Omega)$ is prime ideal of T. Then Ω is bounded T-module if and only if Ω is prime T-module.[5]
Proof. Since every prime module is bounded so by previous corollary we get the proof.
Recall a T-module Ω is called quasi-Dedekind module if every non-zero submodule of Ω is quasi-invertible.[30],[31]
Remark 4.4 Every quasi-Dedekind T-module is bounded module.
Proof. Since every quasi-Dedekind module is prime and hence it is bounded.

In fact, if we think that each proper submodule has at least one element then a quasi-Dedekind T-module can give us a fully bounded T-module which means that every proper submodule is bounded and let see how can so.

Let $A < \Omega$ and assume that $x \in A$, then it is clear that $ann_T(A) \subseteq ann_T(x)$.

Now, let $r \in ann_T(x)$. Since Ω is quasi-Dedekind T-module then A is quasi-invertible submodule and we may define $f: \Omega/A \longrightarrow \Omega$ as $f(x + A) = rx,\ \forall x \in \Omega$, but f=0 so this implies that $rx = 0, \forall x \in \Omega$, and hence , $r\Omega = 0$ so

$r \in ann_T(\Omega) \subseteq ann_T(A)$. Therefore, $ann_T(x) = ann_T(A)$ and A is bounded submodule.

Proposition 4.5 Let Ω be a uniform T-module such that $ann_T(\Omega)$ is prime ideal of T. Then following statements are equivalent:

1- Ω is bounded T-module
2- Ω is fully bounded T-module
3- Ω is prime T-module
4- Ω is quasi-Dedekind T-module

Corollary 4.6 Let Ω be a uniform T-module such that $ann_T(\Omega)$ is prime ideal of T. Then following statements are equivalent:

5- Ω is bounded T-module
6- Ω is fully bounded T-module
7- Ω is quasi-prime T-module
8- Ω is quasi-Dedekind T-module

Recall a T-module Ω is said to monoform if every non-zero submodule of Ω is rational submodule.[8]
Remark 4.7 Every monoform T-module and $ann_T(\Omega)$ is prime ideal of T. Then Ω is fully bounded T-module.
Proof. Since Ω is monoform module, then it is prime and uniform module. Let A be a proper submodule of Ω. Then there exists $0 \neq tx \in A$, then $ann_T(A) \subseteq ann_T(tx)$. Since Ω is prime then it is bounded and hence there exists $x \in \Omega$ such that $ann_T(x) = ann_T(\Omega) \subseteq ann_T(A)$.
Let $r \in ann_T(tx)$ implies that $r(tx) = 0$. Therefore, $r \in ann_T(x)$. Hence $ann_T(tx) = ann_T(A)$. We conclude that A is bounded submodule and since A is arbitrary, then Ω is fully bounded.
Recall a T-module Ω is called Dedekind module if every non-zero submodule of Ω is invertible.[29]
Proposition 4.8 Let Ω be a Dedekind module, then Ω is fully bounded T-module.
Proof. Suppose that Ω is a Dedekind T-module, then by [30] we have for each $x \in \Omega$, $ann_T(x) = ann_T(\Omega)$ and $ann_T(\Omega)$is prime ideal of T. Therefore, Ω is bounded T-module.
Since Ω is Dedekind module, then every non-zero submodule is invertible and by [30] every non-zero submodule is an essential submodule. Now applying proposition (3.6), we get Ω is fully bounded T-module.
Proposition 4.9 Let Ω be a Dedekind T-module, then the following statements are equivalent:

i) Ω is fully bounded T-module.
ii) Ω is prime T-module
iii) Ω is monoform T-module.

Proof. $i \Longrightarrow ii$ it is clear
$ii \Longrightarrow iii$ since every Dedekind T-module is uniform [30] , then Ω is monoform T-module

$iii \Longrightarrow i$ Ω is monofom module , then it is uniform and prime .Thus

Recall a T-module Ω is called compressible module if for every non-zero $\varphi \in Hom(M, A)$ is monomorphism where A is a non-zero submodule of Ω.[13],[34]

Proposition 4.10 Let Ω be a compressible T-module, then Ω is bounded T-module.

Proof. Suppose that Ω is compressible module, then by [13] Ω is prime module and hence Ω is bounded T-module.

Recall a T-module Ω is called scalar module if for every $\varphi \in End(\Omega)$ there exists $r \in T$ such that $\varphi(x) = rx,\ \forall x \in \Omega$.[10]

Corollary 4.11 Let T be an integral domain and Ω is a torsion-free scalar T-module. Then Ω is bounded T-module.

Proof. Let $0 \neq \varphi \in End(\Omega)$ and suppose that $\varphi(x) = \varphi(y)$ for some $x, y \in \Omega$. Thus, there exist $r \in T$ such that $rx = ry$ since Ω is a scalar. Hence, $r(x - y) = 0$ and we know that Ω is torsion-free, we conclude that $x - y = 0$ implies $x = y$.

Therefore, φ is monomorphism and hence Ω is quasi-Dedekind T-module and this implies that Ω is prime so we obtain that Ω is bounded T-module.

Proposition 4.12 Let Ω be a prime T-module and $A, B \leq \Omega$ such that $A \subseteq B \subseteq \Omega$, B is bounded submodule of Ω. Then B is bounded submodule of Ω.

Proof. Suppose that B is bounded submodule, then there exists $x \in B$ such that $ann_T(x) = ann_T(B)$. Since $A \subseteq B$, then $ann_T(B) \subseteq ann_T(A)$ implies that $ann_T(x) \subseteq ann_T(A)$. Now, let $r \in ann_T(A) = ann_T(\Omega) \subseteq \mathrm{ann}_T(B) = ann_T(x)$.

Hence, $ann_T(x) = ann_T(A)$ so A is bounded submodule.

Corollary 4.13 Let Ω be a prime T-module and $A, B \leq \Omega$ such that A is bounded submodule of Ω.

Then $A \cap B$ is bounded submodule of Ω.

Proof. Since $A \cap B \subseteq A \subseteq \Omega$. Then by proposition (4.5) $A \cap B$ is bounded submodule.

Corollary 4.14 Let Ω be a prime T-module and $\{A_i\}_{i=1}^{n}$ be a finite collection of submodules of Ω such that A_i is bounded submodule for $i = 1,2, \dots, n$. Then $\cap_{i=1}^{n} A_i$ is bounded submodule of Ω.

Proof. The proof is by corollary (4.6) and using induction.

Note that in terms of corollary (4.14), the intersection of infinite collection of bounded submodules of Ω is not necessary to be bounded submodule of Ω in general.

Suppose that Z as Z-module, Z is prime Z-module. Since PZ is bounded submodule of Z for each P where P is prime number, but $\bigcap_{P\ is\ prime} PZ = 0$ is not bounded submodule of Z.

5 BOUNDED T-MODULES RELATED FINOTELY ANNIHILATED

In the next discussion, we show the relationship between bounded module and finitely annihilated module.

Recall a T-module Ω is called finitely annihilated T-module if there exists a finite set $\{x_1, x_2, \dots, x_n\}$ where $x_i \in \Omega$,

$i = 1,2,3, \dots, n$ such that $ann_T(\Omega) = \mathrm{ann_T}(\{x_1, x_2, \dots, x_n\})$[23]

Also, there exist an equivalent definition of finitely annihilated module by [24] say that Ω is finitely annihilated module if there exists a finitely generated submodule A of Ω such that

$$ann_T(A) = ann_T(\Omega).$$

Remark 5.1 Every bounded T-module is finitely annihilated T-module.

By [23] Ω is finitely annihilated T-module if and only if there exists an embedding $0 \longrightarrow T/ann_T(\Omega) \longrightarrow \Omega_K$ for direct sum Ω_K of copies of Ω.

Note that if Ω is finitely annihilated T-module and there exists another module M such that $\Omega \subseteq M$

With $ann_T(M) = ann_T(\Omega)$, then M is finitely annihilated T-module.

Remark 5.2 If Ω is finitely annihilated T-module, then any direct or direct product of copies of Ω is finitely annihilated T-module. Also, any submodule of Ω with the descending chain condition on annihilators must be finitely annihilated.

Recall let M be a T-module and we denoted the injective hull of M by E(M) and it is the minimal injective extension of M.[9].

Proposition 5.3 Let Ω be a T-module, then the following statements are equivalent:

1- Ω is bounded T-module
2- E(Ω) is finitely annihilated T-module.
3- E(Ω) is finitely generated as a module over its T-endomorphism ring.

Proof. Since every bounded T-module is finitely annihilated so the proof the proposition is by [23]

Proposition 5.4 Let Ω be a quasi-injective T-module, then the following statements are equivalent:

1- Ω is fully bounded T-module.
2- Every fully invariant submodule of Ω is finitely generated as a right End(Ω)-module.

Proof. Since Ω is fully bounded T-module, then every submodule of Ω is bounded and hence every submodule of Ω is finitely annihilated. Finally, by [23], we get the proof.

We know that every bounded module is finitely annihilated module but the converse is not true so we can raise the following question:

Question 5.5 Under which condition finitely annihilated module gives bounded module?
Note that if the module Ω is multiplication then every finitely annihilated module is finitely generated and if Ω is quasi-injective finitely annihilated module, then Ω is finendo [24] where a T-module is called finendo if it is finitely generated over End(Ω)[24].

Also, there are another versions of bounded module called strongly bounded module and very strongly bounded module which both are stronger than bounded module.

Definition 5.6 A T-module Ω is said to be strongly bounded module if it is bounded and there exists an epimorphism of Ω onto Rx[5].
And here semisimple module play an important role to be bounded module and strongly bounded are equivalent.

Definition 5.7 A T-module Ω is said to be strongly bounded module if it is bounded and Rx is a direct summand of Ω.[5]
It is clear that :
Very strongly bounded ⟹ strongly bounded ⟹ bounded, but the converse is not true in general

Consider $\Omega = 2\mathrm{Z}$ as 2Z-module, then $ann_{2Z}(2Z) = 0 = ann_{2Z}(2)$. Let $\varphi\colon 2Z \longrightarrow 4Z$ define as $\varphi(x) = 2x$, φ is an epimorphism. Thus 2Z is strongly bounded, but it is not very strongly bounded since 4Z is not a direct summand of 2Z [5].

Conclusion

In this systematic review work, we presented a definition of bounded module that introduced by some authors in several ways and we concentrated in some details on Carl Faith definition of bounded module that depend on the annihilator of the module and its elements. Also, we clarify some properties and explain its connection to prime module and other types of modules. Moreover, we introduce some new results that related to bounded modules. Finally, as we see in this paper there are some rich idea that unraveled so other researchers can study bounded module more deeply to discover some results in future.

Conflicts Of Interest

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