



Research Article

Consequence for the Partition (9,9,3)

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ABSTRACT

Let $\mathcal{F}$ be a free module defined on the commutative ring $\mathcal{R}$ with identity. Buchsbaum studied the Weyl module resolution where the Weyl module $\mathcal{K}_{\lambda/\mu}(\mathcal{F})$ is the image of the Weyl map $d'_{\lambda/\mu}(\mathcal{F})$ for the skew-partition λ/μ ; where λ runs over all partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_s)$. There are a number of classical formulas that express the formal character of the representation $\mathcal{L}_{\lambda}(\mathcal{F})$ in terms of standard symmetric polynomials. Such formulas are also valid for the more general representation modules $\{\mathcal{L}_{\lambda/\mu}(\mathcal{F})\}$ associated to skew partition λ/μ ; where $\mu \subseteq \lambda$, where the set of all irreducible polynomial representations of general linear group $GL_n(\mathcal{F})$ of degree n is described by the module $\{\mathcal{L}_{\lambda}(\mathcal{F})\}$

For the partition (9,9,3) by applying the boundary maps we reduction the terms of the characteristic-free of Weyl module resolution to the terms of the Lascoux resolution and prove that the sequence of the reduction terms is exact.

1. INTRODUCTION

The author in [1] described the resolutions $\tilde{B}_{\bullet}$ of Weyl modules by writing down explicit projective resolutions of the two-rowed modules. The existence proof of resolution for the similar problem with an arbitrary number of rows the researchers in [2] gave that. While the existence of resolution of Weyl module whose terms are direct sums of tensor products of divided powers proved by the authors in [3].

Hassan generalized the techniques in [4] for the partitions (3,3,3), and (4,4,3) in [5,6] respectively, also authors in [7-9] studied the cases (8,7,3), (6,6,4;0,0), (7,7,4;0,0).

For the partition (9,9,3) the terms of Weyl module from characteristic-free resolution reduction to the terms of Lascoux resolution and prove that the sequence of these terms is exact in this work.

2. RESOLUTION OF THE CHARACTERISTIC-FREE AND LASCoux

The terms of the resolution for the partition (9,9,3), [4].

$$\begin{aligned} & \text{Res}([9,9;0]) \otimes \mathcal{D}_3 \oplus \sum_{e \geq 0} \underline{Z}_{32}^{(e+1)} \psi \text{Res}([9,9+e+1;e+1]) \otimes \mathcal{D}_{3-e-1} \oplus \\ & \sum_{e_1 \geq 0, e_2 \geq e_1} \underline{Z}_{32}^{(e_2+1)} \psi \underline{Z}_{31}^{(e_1+1)} z \text{Res}([9+e_1+1, 9+e_2+1; e_2-e_1]) \otimes \mathcal{D}_{3-(e_1+e_2+2)} \\ & \text{So} \\ & \sum_{e \geq 0} \underline{Z}_{32}^{(e+1)} \psi \text{Res}([9,9+e+1;e+1]) \otimes \mathcal{D}_{3-e-1} = \underline{Z}_{32} \psi \text{Res}([9,10;1]) \otimes \mathcal{D}_2 \oplus \end{aligned} \quad (1)$$

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$$\underline{Z}_{32}^{(2)} \mathcal{Y} \text{Res}([9,11;2]) \otimes \mathcal{D}_1 \oplus \underline{Z}_{32}^{(3)} \mathcal{Y} \text{Res}([9,12;3]) \otimes \mathcal{D}_0,$$

and

$$\sum_{e_1 \geq 0, e_2 \geq e_1} \underline{Z}_{32}^{(e_2+1)} \mathcal{Y} \underline{Z}_{31}^{(e_1+1)} \mathcal{Z} \text{Res}([9+e_1+1, 9+e_2+1; e_2-e_1]) \otimes \mathcal{D}_{3-(e_1+e_2+2)} = \underline{Z}_{32} \mathcal{Y} \underline{Z}_{31} \mathcal{Z} \text{Res}([10,10;0]) \otimes \mathcal{D}_1 \oplus \underline{Z}_{32}^{(2)} \mathcal{Y} \underline{Z}_{31} \mathcal{Z} \text{Res}([10,11;1]) \otimes \mathcal{D}_0;$$

where $\underline{Z}_{32} \mathcal{Y}$ is the Bar complex:

$$0 \rightarrow \underline{Z}_{32} \mathcal{Y} \xrightarrow{\partial_{\mathcal{Y}}} \underline{Z}_{32} \rightarrow 0,$$

$\underline{Z}_{32}^{(2)} \mathcal{Y}$ is the Bar complex:

$$0 \rightarrow \underline{Z}_{32} \mathcal{Y} \underline{Z}_{32} \mathcal{Y} \xrightarrow{\partial_{\mathcal{Y}}} \underline{Z}_{32}^{(2)} \mathcal{Y} \xrightarrow{\partial_{\mathcal{Y}}} \underline{Z}_{32}^{(2)} \rightarrow 0,$$

$\underline{Z}_{32}^{(3)} \mathcal{Y}$ is the Bar complex:

$$0 \rightarrow \underline{Z}_{32} \mathcal{Y} \underline{Z}_{32} \mathcal{Y} \underline{Z}_{32} \mathcal{Y} \xrightarrow{\partial_{\mathcal{Y}}} \begin{matrix} \underline{Z}_{32}^{(2)} \mathcal{Y} \underline{Z}_{32} \mathcal{Y} \\ \oplus \\ \underline{Z}_{32} \mathcal{Y} \underline{Z}_{32}^{(2)} \mathcal{Y} \end{matrix} \xrightarrow{\partial_{\mathcal{Y}}} \underline{Z}_{32}^{(3)} \mathcal{Y} \xrightarrow{\partial_{\mathcal{Y}}} \underline{Z}_{32}^{(3)} \rightarrow 0,$$

and $\underline{Z}_{31} \mathcal{Z}$ is the Bar complex:

$$0 \rightarrow \underline{Z}_{31} \mathcal{Z} \xrightarrow{\partial_{\mathcal{Z}}} \underline{Z}_{31} \rightarrow 0;$$

where x , $\mathcal{Y}$ and $\mathcal{Z}$ stand for the separator variables, and the boundary map is $\partial_x + \partial_{\mathcal{Y}} + \partial_{\mathcal{Z}}$.

Let $\text{Bar}(\mathcal{M}, \mathcal{A}; \mathcal{S})$ be the free Bar module on the set $\mathcal{S} = \{x, \mathcal{Y}, \mathcal{Z}\}$; where $\mathcal{A}$ is the free associative algebra generated by $\mathcal{Z}_{21}$, $\mathcal{Z}_{32}$, and $\mathcal{Z}_{31}$ and their divided powers with the following relations:

$$\mathcal{Z}_{32}^{(a)} \mathcal{Z}_{31}^{(b)} = \mathcal{Z}_{31}^{(b)} \mathcal{Z}_{32}^{(a)} \quad \text{and} \quad \mathcal{Z}_{21}^{(a)} \mathcal{Z}_{31}^{(b)} = \mathcal{Z}_{31}^{(b)} \mathcal{Z}_{21}^{(a)}.$$

And the module $\mathcal{M}$ is the direct sum of $\mathcal{D}_p \otimes \mathcal{D}_q \otimes \mathcal{D}_r$ for suitable p, q , and r with the action of $\mathcal{Z}_{21}$, $\mathcal{Z}_{32}$, and $\mathcal{Z}_{31}$ and their divided powers.

The terms of the characteristic-free resolution (4.3.1); where $b, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, c_1, c_2 \in \mathbb{Z}^+$ are:

◦ In dimension zero ($\mathcal{X}_0$) we have $\mathcal{D}_9 \otimes \mathcal{D}_9 \otimes \mathcal{D}_3$.

◦ In dimension one ($\mathcal{X}_1$) we have the sum of the following terms:

- $\mathcal{Z}_{21}^{(b)} x \mathcal{D}_{9+b} \otimes \mathcal{D}_{9-b} \otimes \mathcal{D}_3$; where $1 \leq b \leq 9$.
- $\mathcal{Z}_{32}^{(b)} \mathcal{Y} \mathcal{D}_9 \otimes \mathcal{D}_{9+b} \otimes \mathcal{D}_{3-b}$; where $1 \leq b \leq 3$.

◦ In dimension two ($\mathcal{X}_2$) we have the sum of the following terms:

- $\mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{9-|b|} \otimes \mathcal{D}_3$; where $2 \leq |b| = b_1 + b_2 \leq 9$.
- $\mathcal{Z}_{32} \mathcal{Y} \mathcal{Z}_{21}^{(b)} x \mathcal{D}_{9+b} \otimes \mathcal{D}_{10-b} \otimes \mathcal{D}_2$; where $2 \leq b \leq 10$.
- $\mathcal{Z}_{32}^{(2)} \mathcal{Y} \mathcal{Z}_{21}^{(b)} x \mathcal{D}_{9+b} \otimes \mathcal{D}_{11-b} \otimes \mathcal{D}_1$; where $3 \leq b \leq 11$.
- $\mathcal{Z}_{32}^{(3)} \mathcal{Y} \mathcal{Z}_{21}^{(b)} x \mathcal{D}_{9+b} \otimes \mathcal{D}_{12-b} \otimes \mathcal{D}_0$; where $4 \leq b \leq 12$.
- $\mathcal{Z}_{32}^{(b_1)} \mathcal{Y} \mathcal{Z}_{32}^{(b_2)} \mathcal{Y} \mathcal{D}_9 \otimes \mathcal{D}_{12+|b|} \otimes \mathcal{D}_{3-|b|}$; where $2 \leq |b| = b_1 + b_2 \leq 3$.
- $\mathcal{Z}_{32}^{(b)} \mathcal{Y} \mathcal{Z}_{31} \mathcal{Z} \mathcal{D}_{10} \otimes \mathcal{D}_{9+b} \otimes \mathcal{D}_{2-b}$; where $1 \leq b \leq 2$.

◦ In dimension three ($\mathcal{X}_3$) we have the sum of the following terms:

- $\mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{9-|b|} \otimes \mathcal{D}_3$; where $3 \leq |b| = \sum_{i=1}^3 b_i \leq 9$ and $b_1 \geq 1$.
- $\mathcal{Z}_{32} \mathcal{Y} \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{10-|b|} \otimes \mathcal{D}_2$; where $3 \leq |b| = b_1 + b_2 \leq 10$ and $b_1 \geq 2$.
- $\mathcal{Z}_{32}^{(2)} \mathcal{Y} \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{11-|b|} \otimes \mathcal{D}_1$; where $4 \leq |b| = b_1 + b_2 \leq 11$ and $b_1 \geq 3$.
- $\mathcal{Z}_{32} \mathcal{Y} \mathcal{Z}_{32} \mathcal{Y} \mathcal{Z}_{21}^{(b)} x \mathcal{D}_{9+b} \otimes \mathcal{D}_{11-b} \otimes \mathcal{D}_1$; where $3 \leq b \leq 11$.
- $\mathcal{Z}_{32}^{(3)} \mathcal{Y} \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{12-|b|} \otimes \mathcal{D}_0$; where $5 \leq |b| = b_1 + b_2 \leq 12$ and $b_1 \geq 4$.
- $\mathcal{Z}_{32}^{(c_1)} \mathcal{Y} \mathcal{Z}_{32}^{(c_2)} \mathcal{Y} \mathcal{Z}_{21}^{(b)} x \mathcal{D}_{9+b} \otimes \mathcal{D}_{12-b} \otimes \mathcal{D}_0$; where $c_1 + c_2 = 3$ and $4 \leq b \leq 12$.
- $\mathcal{Z}_{32} \mathcal{Y} \mathcal{Z}_{32} \mathcal{Y} \mathcal{Z}_{32} \mathcal{Y} \mathcal{D}_9 \otimes \mathcal{D}_{12} \otimes \mathcal{D}_0$.
- $\mathcal{Z}_{32} \mathcal{Y} \mathcal{Z}_{31} \mathcal{Z} \mathcal{Z}_{21}^{(b)} x \mathcal{D}_{10+b} \otimes \mathcal{D}_{10-b} \otimes \mathcal{D}_1$; where $1 \leq b \leq 10$.

- $Z_{32}^{(2)} \psi Z_{31} z Z_{21}^{(b)} x D_{10+b} \otimes D_{11-b} \otimes D_0$; where $2 \leq b \leq 11$.
- $Z_{32} \psi Z_{32} \psi Z_{31} z D_{10} \otimes D_{11} \otimes D_0$.
- In dimension four ($\mathcal{X}_4$) we have the sum of the following terms:
 - $Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x Z_{21}^{(b_3)} x Z_{21}^{(b_4)} x D_{9+|b|} \otimes D_{9-|b|} \otimes D_3$; where $4 \leq |b| = \sum_{i=1}^4 b_i \leq 9$ and $b_1 \geq 1$.
 - $Z_{32} \psi Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x Z_{21}^{(b_3)} x D_{9+|b|} \otimes D_{10-|b|} \otimes D_2$; where $4 \leq |b| = \sum_{i=1}^3 b_i \leq 10$ and $b_1 \geq 2$.
 - $Z_{32}^{(2)} \psi Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x Z_{21}^{(b_3)} x D_{9+|b|} \otimes D_{11-|b|} \otimes D_1$; where $5 \leq |b| = \sum_{i=1}^3 b_i \leq 11$ and $b_1 \geq 3$.
 - $Z_{32} \psi Z_{32} \psi Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x D_{9+|b|} \otimes D_{11-|b|} \otimes D_1$; where $4 \leq |b| = b_1 + b_2 \leq 11$; and $b_1 \geq 3$.
 - $Z_{32}^{(3)} \psi Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x Z_{21}^{(b_3)} x D_{9+|b|} \otimes D_{12-|b|} \otimes D_0$; where $6 \leq |b| = \sum_{i=1}^3 b_i \leq 12$ and $b_1 \geq 4$.
 - $Z_{32}^{(c_1)} \psi Z_{32}^{(c_2)} \psi Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x D_{9+|b|} \otimes D_{12-|b|} \otimes D_0$; where $c_1 + c_2 = 3$, $5 \leq |b| = b_1 + b_2 \leq 12$ and $b_1 \geq 4$.
 - $Z_{32} \psi Z_{32} \psi Z_{32} \psi Z_{21}^{(b)} x D_{9+b} \otimes D_{12-b} \otimes D_0$; where $4 \leq b \leq 12$.
 - $Z_{32} \psi Z_{31} z Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x D_{10+|b|} \otimes D_{10-|b|} \otimes D_1$; where $2 \leq |b| = b_1 + b_2 \leq 10$ and $b_1 \geq 1$.
 - $Z_{32}^{(2)} \psi Z_{31} z Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x D_{10+|b|} \otimes D_{11-|b|} \otimes D_0$; where $3 \leq |b| = b_1 + b_2 \leq 11$ and $b_1 \geq 2$.
 - $Z_{32} \psi Z_{32} \psi Z_{31} z Z_{21}^{(b)} x D_{10+b} \otimes D_{11-b} \otimes D_0$; where $2 \leq b \leq 11$.
- In dimension five ($\mathcal{X}_5$) we have the sum of the following terms:
 - $Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x Z_{21}^{(b_3)} x Z_{21}^{(b_4)} x Z_{21}^{(b_5)} x D_{9+|b|} \otimes D_{9-|b|} \otimes D_3$; where $5 \leq |b| = \sum_{i=1}^5 b_i \leq 9$ and $b_1 \geq 1$.
 - $Z_{32} \psi Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x Z_{21}^{(b_3)} x Z_{21}^{(b_4)} x D_{9+|b|} \otimes D_{10-|b|} \otimes D_2$; where $5 \leq |b| = \sum_{i=1}^4 b_i \leq 10$ and $b_1 \geq 2$.
 - $Z_{32}^{(2)} \psi Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x Z_{21}^{(b_3)} x Z_{21}^{(b_4)} x D_{9+|b|} \otimes D_{11-|b|} \otimes D_1$; where $6 \leq |b| = \sum_{i=1}^4 b_i \leq 11$ and $b_1 \geq 3$.
 - $Z_{32} \psi Z_{32} \psi Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x Z_{21}^{(b_3)} x D_{9+|b|} \otimes D_{11-|b|} \otimes D_1$; where $5 \leq |b| = \sum_{i=1}^3 b_i \leq 11$ and $b_1 \geq 3$.
 - $Z_{32}^{(3)} \psi Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x Z_{21}^{(b_3)} x Z_{21}^{(b_4)} x D_{9+|b|} \otimes D_{12-|b|} \otimes D_0$; where $7 \leq |b| = \sum_{i=1}^4 b_i \leq 12$ and $b_1 \geq 4$.
 - $Z_{32}^{(c_1)} \psi Z_{32}^{(c_2)} \psi Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x Z_{21}^{(b_3)} x D_{9+|b|} \otimes D_{12-|b|} \otimes D_0$; where $c_1 + c_2 = 3$, $6 \leq |b| = \sum_{i=1}^3 b_i \leq 12$ and $b_1 \geq 4$.
 - $Z_{32} \psi Z_{32} \psi Z_{32} \psi Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x D_{9+|b|} \otimes D_{12-|b|} \otimes D_0$; where $5 \leq |b| = b_1 + b_2 \leq 12$ and $b_1 \geq 4$.
 - $Z_{32} \psi Z_{31} z Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x Z_{21}^{(b_3)} x D_{10+|b|} \otimes D_{10-|b|} \otimes D_1$; where $3 \leq |b| = \sum_{i=1}^3 b_i \leq 10$ and $b_1 \geq 1$.
 - $Z_{32}^{(2)} \psi Z_{31} z Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x Z_{21}^{(b_3)} x D_{10+|b|} \otimes D_{11-|b|} \otimes D_0$; where $4 \leq |b| = \sum_{i=1}^3 b_i \leq 11$ and $b_1 \geq 2$.
 - $Z_{32} \psi Z_{32} \psi Z_{31} z Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x D_{10+|b|} \otimes D_{11-|b|} \otimes D_0$; where $3 \leq |b| = b_1 + b_2 \leq 11$ and $b_1 \geq 2$.
- In dimension six ($\mathcal{X}_6$) we have the sum of the following terms:
 - $Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x Z_{21}^{(b_3)} x Z_{21}^{(b_4)} x Z_{21}^{(b_5)} x Z_{21}^{(b_6)} x D_{9+|b|} \otimes D_{9-|b|} \otimes D_3$; where $6 \leq |b| = \sum_{i=1}^6 b_i \leq 9$ and $b_1 \geq 1$.
 - $Z_{32} \psi Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x Z_{21}^{(b_3)} x Z_{21}^{(b_4)} x Z_{21}^{(b_5)} x D_{9+|b|} \otimes D_{10-|b|} \otimes D_2$; where $6 \leq |b| = \sum_{i=1}^5 b_i \leq 10$ and $b_1 \geq 2$.
 - $Z_{32}^{(2)} \psi Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x Z_{21}^{(b_3)} x Z_{21}^{(b_4)} x Z_{21}^{(b_5)} x D_{9+|b|} \otimes D_{11-|b|} \otimes D_1$; where $7 \leq |b| = \sum_{i=1}^5 b_i \leq 11$ and $b_1 \geq 3$.
 - $Z_{32} \psi Z_{32} \psi Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x Z_{21}^{(b_3)} x Z_{21}^{(b_4)} x D_{9+|b|} \otimes D_{11-|b|} \otimes D_1$; where $6 \leq |b| = \sum_{i=1}^4 b_i \leq 11$ and $b_1 \geq 3$.
 - $Z_{32}^{(3)} \psi Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x Z_{21}^{(b_3)} x Z_{21}^{(b_4)} x Z_{21}^{(b_5)} x D_{9+|b|} \otimes D_{12-|b|} \otimes D_0$; where $8 \leq |b| = \sum_{i=1}^5 b_i \leq 12$ and $b_1 \geq 4$.
 - $Z_{32}^{(c_1)} \psi Z_{32}^{(c_2)} \psi Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x Z_{21}^{(b_3)} x Z_{21}^{(b_4)} x D_{9+|b|} \otimes D_{12-|b|} \otimes D_0$; where $c_1 + c_2 = 3$, $7 \leq |b| = \sum_{i=1}^4 b_i \leq 12$ and $b_1 \geq 4$.
 - $Z_{32} \psi Z_{32} \psi Z_{32} \psi Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x Z_{21}^{(b_3)} x D_{9+|b|} \otimes D_{12-|b|} \otimes D_0$; where $6 \leq |b| = \sum_{i=1}^3 b_i \leq 12$ and $b_1 \geq 4$.
 - $Z_{32} \psi Z_{31} z Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x Z_{21}^{(b_3)} x Z_{21}^{(b_4)} x D_{10+|b|} \otimes D_{10-|b|} \otimes D_1$; where $4 \leq |b| = \sum_{i=1}^4 b_i \leq 10$ and $b_1 \geq 1$.
 - $Z_{32}^{(2)} \psi Z_{31} z Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x Z_{21}^{(b_3)} x Z_{21}^{(b_4)} x D_{10+|b|} \otimes D_{11-|b|} \otimes D_0$; where $5 \leq |b| = \sum_{i=1}^4 b_i \leq 11$ and $b_1 \geq 2$.
 - $Z_{32} \psi Z_{32} \psi Z_{31} z Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x Z_{21}^{(b_3)} x D_{10+|b|} \otimes D_{11-|b|} \otimes D_0$; where $4 \leq |b| = \sum_{i=1}^3 b_i \leq 11$ and $b_1 \geq 2$.
- In dimension seven ($\mathcal{X}_7$) we have the sum of the following terms:
 - $Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x Z_{21}^{(b_3)} x Z_{21}^{(b_4)} x Z_{21}^{(b_5)} x Z_{21}^{(b_6)} x Z_{21}^{(b_7)} x D_{9+|b|} \otimes D_{9-|b|} \otimes D_3$; where $7 \leq |b| = \sum_{i=1}^7 b_i \leq 9$ and $b_1 \geq 1$.
 - $Z_{32} \psi Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x Z_{21}^{(b_3)} x Z_{21}^{(b_4)} x Z_{21}^{(b_5)} x Z_{21}^{(b_6)} x D_{9+|b|} \otimes D_{10-|b|} \otimes D_2$; where $7 \leq |b| = \sum_{i=1}^6 b_i \leq 10$ and $b_1 \geq 2$.
 - $Z_{32}^{(2)} \psi Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x Z_{21}^{(b_3)} x Z_{21}^{(b_4)} x Z_{21}^{(b_5)} x Z_{21}^{(b_6)} x D_{9+|b|} \otimes D_{11-|b|} \otimes D_1$; where $8 \leq |b| = \sum_{i=1}^6 b_i \leq 11$ and $b_1 \geq 3$.
 - $Z_{32} \psi Z_{32} \psi Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x Z_{21}^{(b_3)} x Z_{21}^{(b_4)} x Z_{21}^{(b_5)} x D_{9+|b|} \otimes D_{11-|b|} \otimes D_1$; where $7 \leq |b| = \sum_{i=1}^5 b_i \leq 11$ and $b_1 \geq 3$.
 - $Z_{32}^{(3)} \psi Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x Z_{21}^{(b_3)} x Z_{21}^{(b_4)} x Z_{21}^{(b_5)} x Z_{21}^{(b_6)} x D_{9+|b|} \otimes D_{12-|b|} \otimes D_0$; where $9 \leq |b| = \sum_{i=1}^6 b_i \leq 12$ and $b_1 \geq 4$.

where $c_1 + c_2 = 3$, $11 \leq |b| = \sum_{i=1}^8 b_i \leq 12$ and $b_1 \geq 4$.

- $Z_{32} y Z_{32} y Z_{32} y Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x Z_{21}^{(b_3)} x Z_{21}^{(b_4)} x Z_{21}^{(b_5)} x Z_{21}^{(b_6)} x Z_{21}^{(b_7)} x D_{9+|b|} \otimes D_{12-|b|} \otimes D_0$;
where $10 \leq |b| = \sum_{i=1}^7 b_i \leq 12$ and $b_1 \geq 4$.
- $Z_{32} y Z_{31} z Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x Z_{21}^{(b_3)} x Z_{21}^{(b_4)} x Z_{21}^{(b_5)} x Z_{21}^{(b_6)} x Z_{21}^{(b_7)} x Z_{21}^{(b_8)} x D_{10+|b|} \otimes D_{10-|b|} \otimes D_1$;
where $8 \leq |b| = \sum_{i=1}^8 b_i \leq 10$ and $b_1 \geq 1$.
- $Z_{32}^{(2)} y Z_{31} z Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x Z_{21}^{(b_3)} x Z_{21}^{(b_4)} x Z_{21}^{(b_5)} x Z_{21}^{(b_6)} x Z_{21}^{(b_7)} x Z_{21}^{(b_8)} x D_{10+|b|} \otimes D_{11-|b|} \otimes D_0$;
where $9 \leq |b| = \sum_{i=1}^6 b_i \leq 11$ and $b_1 \geq 2$.
- $Z_{32} y Z_{32} y Z_{31} z Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x Z_{21}^{(b_3)} x Z_{21}^{(b_4)} x Z_{21}^{(b_5)} x Z_{21}^{(b_6)} x Z_{21}^{(7)} x D_{10+|b|} \otimes D_{11-|b|} \otimes D_0$;
where $8 \leq |b| = \sum_{i=1}^7 b_i \leq 11$ and $b_1 \geq 2$.

◦ In dimension eleven ($\mathcal{X}_{11}$) we have the sum of the following terms:

- $Z_{32} y Z_{32} y Z_{21}^{(3)} x Z_{21} x Z_{21} x Z_{21} x Z_{21} x Z_{21} x Z_{21} x Z_{21} x D_{20} \otimes D_0 \otimes D_1$.
- $Z_{32}^{(c_1)} y Z_{32}^{(c_2)} y Z_{21}^{(4)} x Z_{21} x Z_{21} x Z_{21} x Z_{21} x Z_{21} x Z_{21} x Z_{21} x D_{9+|b|} \otimes D_{12-|b|} \otimes D_0$; where $c_1 + c_2 = 3$.
- $Z_{32} y Z_{32} y Z_{32} y Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x Z_{21}^{(b_3)} x Z_{21}^{(b_4)} x Z_{21}^{(b_5)} x Z_{21}^{(b_6)} x Z_{21}^{(b_7)} x Z_{21}^{(b_8)} x D_{9+|b|} \otimes D_{12-|b|} \otimes D_0$;
where $11 \leq |b| = \sum_{i=1}^8 b_i \leq 12$ and $b_1 \geq 4$.
- $Z_{32} y Z_{31} z Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x Z_{21}^{(b_3)} x Z_{21}^{(b_4)} x Z_{21}^{(b_5)} x Z_{21}^{(b_6)} x Z_{21}^{(b_7)} x Z_{21}^{(8)} x Z_{21}^{(b_9)} x D_{10+|b|} \otimes D_{10-|b|} \otimes D_1$;
where $9 \leq |b| = \sum_{i=1}^9 b_i \leq 10$ and $b_1 \geq 1$.
- $Z_{32}^{(2)} y Z_{31} z Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x Z_{21}^{(b_3)} x Z_{21}^{(b_4)} x Z_{21}^{(b_5)} x Z_{21}^{(b_6)} x Z_{21}^{(b_7)} x Z_{21}^{(b_8)} x Z_{21}^{(b_9)} x D_{10+|b|} \otimes D_{11-|b|} \otimes D_0$;
where $10 \leq |b| = \sum_{i=1}^9 b_i \leq 11$ and $b_1 \geq 2$.
- $Z_{32} y Z_{32} y Z_{31} z Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x Z_{21}^{(b_3)} x Z_{21}^{(b_4)} x Z_{21}^{(b_5)} x Z_{21}^{(b_6)} x Z_{21}^{(b_7)} x Z_{21}^{(b_8)} x D_{10+|b|} \otimes D_{11-|b|} \otimes D_0$;
where $9 \leq |b| = \sum_{i=1}^8 b_i \leq 11$ and $b_1 \geq 2$.

◦ In dimension twelve ($\mathcal{X}_{12}$) we have the sum of the following terms:

- $Z_{32} y Z_{32} y Z_{32} y Z_{21}^{(4)} x Z_{21} x Z_{21} x Z_{21} x Z_{21} x Z_{21} x Z_{21} x Z_{21} x D_{21} \otimes D_0 \otimes D_0$.
- $Z_{32} y Z_{31} z Z_{21} x Z_{21} x Z_{21} x Z_{21} x Z_{21} x Z_{21} x Z_{21} x Z_{21} x D_{20} \otimes D_0 \otimes D_1$.
- $Z_{32}^{(2)} y Z_{31} z Z_{21} x Z_{21} x Z_{21} x Z_{21} x Z_{21} x Z_{21} x Z_{21} x Z_{21} x D_{21} \otimes D_0 \otimes D_0$.
- $Z_{32} y Z_{32} y Z_{31} z Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x Z_{21}^{(b_3)} x Z_{21}^{(b_4)} x Z_{21}^{(b_5)} x Z_{21}^{(b_6)} x Z_{21}^{(b_7)} x Z_{21}^{(b_8)} x Z_{21}^{(b_9)} x D_{10-|b|} \otimes D_{11-|b|} \otimes D_0$;
where $10 \leq |b| = \sum_{i=1}^9 b_i \leq 11$ and $b_1 \geq 2$.

Finally, in dimension thirteen ($\mathcal{X}_{13}$) we have:

- $Z_{32}^{(1)} y Z_{32}^{(1)} y Z_{31}^{(1)} z Z_{21}^{(2)} x Z_{21}^{(1)} x Z_{21}^{(1)} x Z_{21}^{(1)} x Z_{21}^{(1)} x Z_{21}^{(1)} x Z_{21}^{(1)} x Z_{21}^{(1)} x D_{21} \otimes D_0 \otimes D_0$.

The terms of the Lascoux complex are obtained by the determinantal expansion of the Jacobi-Trudi matrix of the partition [1]. The positions of the terms of the complex are determined by the length of the permutation to which they correspond, [4].

In the case of the partition (9,9,3) we get the following matrix:

$$\begin{bmatrix} \mathcal{D}_9 \mathcal{F} & \mathcal{D}_8 \mathcal{F} & \mathcal{D}_1 \mathcal{F} \\ \mathcal{D}_{10} \mathcal{F} & \mathcal{D}_9 \mathcal{F} & \mathcal{D}_2 \mathcal{F} \\ \mathcal{D}_{11} \mathcal{F} & \mathcal{D}_{10} \mathcal{F} & \mathcal{D}_3 \mathcal{F} \end{bmatrix}$$

Then the Lascoux complex has the correspondence between its terms as follows:

$$\mathcal{D}_9 \mathcal{F} \otimes \mathcal{D}_9 \mathcal{F} \otimes \mathcal{D}_3 \mathcal{F} \leftrightarrow \text{identity}.$$

$$\mathcal{D}_{10} \mathcal{F} \otimes \mathcal{D}_8 \mathcal{F} \otimes \mathcal{D}_3 \mathcal{F} \leftrightarrow (12).$$

$$\mathcal{D}_9 \mathcal{F} \otimes \mathcal{D}_{10} \mathcal{F} \otimes \mathcal{D}_2 \mathcal{F} \leftrightarrow (23).$$

$$\mathcal{D}_{10} \mathcal{F} \otimes \mathcal{D}_{10} \mathcal{F} \otimes \mathcal{D}_1 \mathcal{F} \leftrightarrow (123).$$

$$\mathcal{D}_{11} \mathcal{F} \otimes \mathcal{D}_8 \mathcal{F} \otimes \mathcal{D}_2 \mathcal{F} \leftrightarrow (132).$$

$$\mathcal{D}_{11} \mathcal{F} \otimes \mathcal{D}_9 \mathcal{F} \otimes \mathcal{D}_1 \mathcal{F} \leftrightarrow (13).$$

Thus the resolution of Lascoux in the case of the partition (9,9,3) has the formulation:

$$\mathcal{D}_{11} \mathcal{F} \otimes \mathcal{D}_9 \mathcal{F} \otimes \mathcal{D}_1 \mathcal{F} \longrightarrow \bigoplus \begin{matrix} \mathcal{D}_{11} \mathcal{F} \otimes \mathcal{D}_8 \mathcal{F} \otimes \mathcal{D}_2 \mathcal{F} \\ \mathcal{D}_{10} \mathcal{F} \otimes \mathcal{D}_8 \mathcal{F} \otimes \mathcal{D}_3 \mathcal{F} \end{matrix} \longrightarrow \bigoplus \begin{matrix} \mathcal{D}_{10} \mathcal{F} \otimes \mathcal{D}_9 \mathcal{F} \otimes \mathcal{D}_2 \mathcal{F} \\ \mathcal{D}_9 \mathcal{F} \otimes \mathcal{D}_{10} \mathcal{F} \otimes \mathcal{D}_3 \mathcal{F} \end{matrix} \longrightarrow \mathcal{D}_9 \mathcal{F} \otimes \mathcal{D}_9 \mathcal{F} \otimes \mathcal{D}_3 \mathcal{F}$$

$$\mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_1\mathcal{F} \quad \mathcal{D}_9\mathcal{F} \otimes \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_2\mathcal{F}$$

3. THE CONSEQUENCE

As in [4], we exhibit the terms of the complex (2.1) as:

$$\mathcal{X}_0 = \mathcal{L}_0 = \mathcal{M}_0,$$

$$\mathcal{X}_1 = \mathcal{L}_1 \oplus \mathcal{M}_1,$$

$$\mathcal{X}_2 = \mathcal{L}_2 \oplus \mathcal{M}_2,$$

$$\mathcal{X}_3 = \mathcal{L}_3 \oplus \mathcal{M}_3,$$

$$\mathcal{X}_j = \mathcal{M}_j \text{ ; for } j = 4, 5, \dots, 13,$$

where $\mathcal{L}_e$ are the sum of the Lascoux terms and $\mathcal{M}_e$ are the sum of the others.

Now, we define the map $\sigma_1: \mathcal{M}_1 \longrightarrow \mathcal{L}_1$ such that

$$\delta_{\mathcal{L}_1\mathcal{L}_0} \circ \sigma_1 = \delta_{\mathcal{M}_1\mathcal{M}_0} \quad (2)$$

As follows:

- $Z_{21}^{(2)}x(v) \mapsto \frac{1}{2} Z_{21}x\partial_{21}(v)$; where $v \in \mathcal{D}_{11} \otimes \mathcal{D}_7 \otimes \mathcal{D}_3$.
- $Z_{21}^{(3)}x(v) \mapsto \frac{1}{3} Z_{21}x\partial_{21}^{(2)}(v)$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_6 \otimes \mathcal{D}_3$.
- $Z_{21}^{(4)}x(v) \mapsto \frac{1}{4} Z_{21}x\partial_{21}^{(3)}(v)$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_5 \otimes \mathcal{D}_3$.
- $Z_{21}^{(5)}x(v) \mapsto \frac{1}{5} Z_{21}x\partial_{21}^{(4)}(v)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_4 \otimes \mathcal{D}_3$.
- $Z_{21}^{(6)}x(v) \mapsto \frac{1}{6} Z_{21}x\partial_{21}^{(5)}(v)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_3 \otimes \mathcal{D}_3$.
- $Z_{21}^{(7)}x(v) \mapsto \frac{1}{7} Z_{21}x\partial_{21}^{(6)}(v)$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_2 \otimes \mathcal{D}_3$.
- $Z_{21}^{(8)}x(v) \mapsto \frac{1}{8} Z_{21}x\partial_{21}^{(7)}(v)$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_1 \otimes \mathcal{D}_3$.
- $Z_{21}^{(9)}x(v) \mapsto \frac{1}{8} Z_{21}x\partial_{21}^{(8)}(v)$; where $v \in \mathcal{D}_{18} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$.
- $Z_{32}^{(2)}y(v) \mapsto \frac{1}{2} Z_{32}y\partial_{32}(v)$; where $v \in \mathcal{D}_9 \otimes \mathcal{D}_{11} \otimes \mathcal{D}_1$.
- $Z_{32}^{(3)}y(v) \mapsto \frac{1}{3} Z_{32}y\partial_{32}^{(2)}(v)$; where $v \in \mathcal{D}_9 \otimes \mathcal{D}_{12} \otimes \mathcal{D}_0$.

It is clear that σ_1 satisfies (4.4.1), then we can define:

$$\partial_1: \mathcal{L}_1 \longrightarrow \mathcal{L}_0 \text{ as } \partial_1 = \delta_{\mathcal{L}_1\mathcal{L}_0}$$

At this point, we are in a position to define

$$\partial_2: \mathcal{L}_2 \longrightarrow \mathcal{L}_1 \text{ by } \partial_2 = \delta_{\mathcal{L}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{L}_2\mathcal{M}_1}$$

Lemma (3.1):

The composition $\partial_1\partial_2$ equal to zero.

Proof:

$$\begin{aligned} \partial_1\partial_2(a) &= \delta_{\mathcal{L}_1\mathcal{L}_0} \circ (\delta_{\mathcal{L}_2\mathcal{L}_1}(a) + (\sigma_1 \circ \delta_{\mathcal{L}_2\mathcal{M}_1})(a)) \\ &= \delta_{\mathcal{L}_1\mathcal{L}_0} \circ \delta_{\mathcal{L}_2\mathcal{L}_1}(a) + \delta_{\mathcal{L}_1\mathcal{L}_0} \circ (\sigma_1 \circ \delta_{\mathcal{L}_2\mathcal{M}_1})(a). \end{aligned}$$

But $\delta_{\mathcal{L}_1\mathcal{L}_0} \circ \sigma_1 = \delta_{\mathcal{M}_1\mathcal{M}_0}$ then we get:

$$\partial_1\partial_2(a) = \delta_{\mathcal{L}_1\mathcal{L}_0} \circ \delta_{\mathcal{L}_2\mathcal{L}_1}(a) + \delta_{\mathcal{M}_1\mathcal{M}_0} \circ \delta_{\mathcal{L}_2\mathcal{M}_1}(a).$$

By properties of the boundary map δ we get:

$$\partial_1\partial_2 = 0$$

We need to define the map $\sigma_2: \mathcal{M}_2 \longrightarrow \mathcal{L}_2$ such that

$$\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1} = (\delta_{\mathcal{L}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{L}_2\mathcal{M}_1}) \circ \sigma_2 \quad (3)$$

As follows:

- $Z_{21}xZ_{21}x(v) \mapsto 0$; where $v \in \mathcal{D}_{11} \otimes \mathcal{D}_7 \otimes \mathcal{D}_3$.
- $Z_{21}^{(2)}xZ_{21}x(v) \mapsto 0$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_6 \otimes \mathcal{D}_3$.
- $Z_{21}xZ_{21}^{(2)}x(v) \mapsto 0$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_6 \otimes \mathcal{D}_3$.
- $Z_{21}^{(3)}xZ_{21}x(v) \mapsto 0$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_5 \otimes \mathcal{D}_3$.
- $Z_{21}xZ_{21}^{(3)}x(v) \mapsto 0$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_5 \otimes \mathcal{D}_3$.

- [illegible]

- $Z_{32}^{(2)} \psi Z_{21}^{(6)} x(v) \mapsto \frac{1}{60} Z_{32} \psi Z_{21}^{(2)} x \partial_{21}^{(3)} \partial_{31}(v) - \frac{1}{6} Z_{32} \psi Z_{31} z \partial_{21}^{(5)}(v)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_5 \otimes \mathcal{D}_1$.
- $Z_{32}^{(2)} \psi Z_{21}^{(7)} x(v) \mapsto \frac{1}{105} Z_{32} \psi Z_{21}^{(2)} x \partial_{21}^{(4)} \partial_{31}(v) - \frac{1}{7} Z_{32} \psi Z_{31} z \partial_{21}^{(6)}(v)$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_4 \otimes \mathcal{D}_1$.
- $Z_{32}^{(2)} \psi Z_{21}^{(8)} x(v) \mapsto \frac{1}{168} Z_{32} \psi Z_{21}^{(2)} x \partial_{21}^{(5)} \partial_{31}(v) - \frac{1}{8} Z_{32} \psi Z_{31} z \partial_{21}^{(7)}(v)$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_3 \otimes \mathcal{D}_1$.
- $Z_{32}^{(2)} \psi Z_{21}^{(9)} x(v) \mapsto \frac{1}{252} Z_{32} \psi Z_{21}^{(2)} x \partial_{21}^{(6)} \partial_{31}(v) - \frac{1}{9} Z_{32} \psi Z_{31} z \partial_{21}^{(8)}(v)$; where $v \in \mathcal{D}_{18} \otimes \mathcal{D}_2 \otimes \mathcal{D}_1$.
- $Z_{32}^{(2)} \psi Z_{21}^{(10)} x(v) \mapsto \frac{1}{360} Z_{32} \psi Z_{21}^{(2)} x \partial_{21}^{(7)} \partial_{31}(v) - \frac{1}{10} Z_{32} \psi Z_{31} z \partial_{21}^{(9)}(v)$; where $v \in \mathcal{D}_{19} \otimes \mathcal{D}_1 \otimes \mathcal{D}_1$.
- $Z_{32}^{(2)} \psi Z_{21}^{(11)} x(v) \mapsto \frac{1}{495} Z_{32} \psi Z_{21}^{(2)} x \partial_{21}^{(8)} \partial_{31}(v) - \frac{1}{9} Z_{32} \psi Z_{31} z \partial_{21}^{(10)}(v)$; where $v \in \mathcal{D}_{20} \otimes \mathcal{D}_0 \otimes \mathcal{D}_1$.
- $Z_{32}^{(2)} \psi Z_{32} \psi(v) \mapsto 0$; where $v \in \mathcal{D}_9 \otimes \mathcal{D}_{12} \otimes \mathcal{D}_0$.
- $Z_{32} \psi Z_{32}^{(2)} \psi(v) \mapsto 0$; where $v \in \mathcal{D}_9 \otimes \mathcal{D}_{12} \otimes \mathcal{D}_0$.
- $Z_{32}^{(3)} \psi Z_{21}^{(4)} x(v) \mapsto \frac{1}{3} Z_{32} \psi Z_{21}^{(2)} x \partial_{31}^{(2)}(v) - \frac{1}{6} Z_{32} \psi Z_{21}^{(2)} x \partial_{21}^{(2)} \partial_{32}^{(2)}(v) - \frac{1}{3} Z_{32} \psi Z_{31} z \partial_{21}^{(3)} \partial_{32}(v)$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_8 \otimes \mathcal{D}_0$.
- $Z_{32}^{(3)} \psi Z_{21}^{(5)} x(v) \mapsto \frac{1}{9} Z_{32} \psi Z_{21}^{(2)} x \partial_{21} \partial_{31}^{(2)}(v) - \frac{7}{90} Z_{32} \psi Z_{21}^{(2)} x \partial_{21}^{(3)} \partial_{32}^{(2)}(v) - \frac{2}{9} Z_{32} \psi Z_{31} z \partial_{21}^{(4)} \partial_{32}(v)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_7 \otimes \mathcal{D}_0$.
- $Z_{32}^{(3)} \psi Z_{21}^{(6)} x(v) \mapsto \frac{1}{18} Z_{32} \psi Z_{21}^{(2)} x \partial_{21}^{(2)} \partial_{31}^{(2)}(v) - \frac{2}{45} Z_{32} \psi Z_{21}^{(2)} x \partial_{21}^{(4)} \partial_{32}^{(2)}(v) - \frac{1}{6} Z_{32} \psi Z_{31} z \partial_{21}^{(5)} \partial_{32}(v)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_6 \otimes \mathcal{D}_0$.
- $Z_{32}^{(3)} \psi Z_{21}^{(7)} x(v) \mapsto \frac{1}{30} Z_{32} \psi Z_{21}^{(2)} x \partial_{21}^{(3)} \partial_{31}^{(2)}(v) - \frac{1}{35} Z_{32} \psi Z_{21}^{(2)} x \partial_{21}^{(5)} \partial_{32}^{(2)}(v) - \frac{2}{15} Z_{32} \psi Z_{31} z \partial_{21}^{(6)} \partial_{32}(v)$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_5 \otimes \mathcal{D}_0$.
- $Z_{32}^{(3)} \psi Z_{21}^{(8)} x(v) \mapsto \frac{1}{45} Z_{32} \psi Z_{21}^{(2)} x \partial_{21}^{(4)} \partial_{31}^{(2)}(v) - \frac{5}{252} Z_{32} \psi Z_{21}^{(2)} x \partial_{21}^{(6)} \partial_{32}^{(2)}(v) - \frac{1}{9} Z_{32} \psi Z_{31} z \partial_{21}^{(7)} \partial_{32}(v)$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_4 \otimes \mathcal{D}_0$.
- $Z_{32}^{(3)} \psi Z_{21}^{(9)} x(v) \mapsto \frac{1}{63} Z_{32} \psi Z_{21}^{(2)} x \partial_{21}^{(5)} \partial_{31}^{(2)}(v) - \frac{11}{756} Z_{32} \psi Z_{21}^{(2)} x \partial_{21}^{(7)} \partial_{32}^{(2)}(v) - \frac{2}{21} Z_{32} \psi Z_{31} z \partial_{21}^{(8)} \partial_{32}(v)$; where $v \in \mathcal{D}_{18} \otimes \mathcal{D}_3 \otimes \mathcal{D}_0$.
- $Z_{32}^{(3)} \psi Z_{21}^{(10)} x(v) \mapsto \frac{1}{84} Z_{32} \psi Z_{21}^{(2)} x \partial_{21}^{(6)} \partial_{31}^{(2)}(v) - \frac{1}{90} Z_{32} \psi Z_{21}^{(2)} x \partial_{21}^{(8)} \partial_{32}^{(2)}(v) - \frac{1}{12} Z_{32} \psi Z_{31} z \partial_{21}^{(9)} \partial_{32}(v)$; where $v \in \mathcal{D}_{19} \otimes \mathcal{D}_2 \otimes \mathcal{D}_0$.
- $Z_{32}^{(3)} \psi Z_{21}^{(11)} x(v) \mapsto \frac{1}{108} Z_{32} \psi Z_{21}^{(2)} x \partial_{21}^{(7)} \partial_{31}^{(2)}(v) + \frac{1}{135} Z_{32} \psi Z_{21}^{(2)} x \partial_{21}^{(8)} \partial_{32} \partial_{31}(v)$; where $v \in \mathcal{D}_{20} \otimes \mathcal{D}_1 \otimes \mathcal{D}_0$.
- $Z_{32}^{(3)} \psi Z_{21}^{(12)} x(v) \mapsto \frac{1}{135} Z_{32} \psi Z_{21}^{(2)} x \partial_{21}^{(8)} \partial_{31}^{(2)}(v)$; where $v \in \mathcal{D}_{21} \otimes \mathcal{D}_0 \otimes \mathcal{D}_0$.
- $Z_{32}^{(2)} \psi Z_{31} z(v) \mapsto \frac{1}{3} Z_{32} \psi Z_{31} z \partial_{32}(v)$; where $v \in \mathcal{D}_{10} \otimes \mathcal{D}_{11} \otimes \mathcal{D}_0$.

Proposition (3.2):

The map σ_2 defined above satisfies (3.2).

Proof:

We can see that for some terms:

$$\begin{aligned} & \bullet (\delta_{\mathcal{M}_2 \mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2 \mathcal{M}_1})(Z_{21} x Z_{21} x(v)); \text{ where } v \in \mathcal{D}_{11} \otimes \mathcal{D}_7 \otimes \mathcal{D}_3 \\ &= \sigma_1 \left(2 Z_{21}^{(2)} x(v) \right) - Z_{21} x \partial_{21}(v) = \frac{2}{2} Z_{21} x \partial_{21}(v) - Z_{21} x \partial_{21}(v) = 0. \end{aligned}$$

$$\begin{aligned} & \bullet (\delta_{\mathcal{M}_2 \mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2 \mathcal{M}_1}) \left(Z_{21}^{(2)} x Z_{21} x(v) \right); \text{ where } v \in \mathcal{D}_{12} \otimes \mathcal{D}_6 \otimes \mathcal{D}_3 \\ &= \sigma_1 \left(3 Z_{21}^{(3)} x(v) - Z_{21}^{(2)} x \partial_{21}(v) \right) = Z_{21} x \partial_{21}^{(2)}(v) - \frac{2}{2} Z_{21} x \partial_{21}^{(2)}(v) = 0. \end{aligned}$$

$$\begin{aligned} & \bullet (\delta_{\mathcal{M}_2 \mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2 \mathcal{M}_1}) \left(Z_{21}^{(2)} x Z_{21}^{(3)} x(v) \right); \text{ where } v \in \mathcal{D}_{14} \otimes \mathcal{D}_4 \otimes \mathcal{D}_3 \\ &= \sigma_1 \left(10 Z_{21}^{(5)} x(v) - Z_{21}^{(2)} x \partial_{21}^{(3)}(v) \right) = 2 Z_{21} x \partial_{21}^{(4)}(v) - \frac{4}{2} Z_{21} x \partial_{21}^{(4)}(v) = 0. \end{aligned}$$

$$\begin{aligned} & \bullet (\delta_{\mathcal{M}_2 \mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2 \mathcal{M}_1}) \left(Z_{32} \psi Z_{21}^{(3)} x(v) \right); \text{ where } v \in \mathcal{D}_{12} \otimes \mathcal{D}_7 \otimes \mathcal{D}_2 \\ &= \sigma_1 \left(Z_{21}^{(3)} x \partial_{32}(v) + Z_{21}^{(2)} x \partial_{31}(v) \right) - Z_{32} \psi \partial_{21}^{(3)}(v) = \frac{1}{3} Z_{21} x \partial_{21}^{(2)} \partial_{32}(v) + \frac{1}{2} Z_{21} x \partial_{21} \partial_{31}(v) - Z_{32} \psi \partial_{21}^{(3)}(v). \end{aligned}$$

And

$$(\delta_{\mathcal{L}_2 \mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{L}_2 \mathcal{M}_1}) \left(\frac{1}{3} Z_{32} \psi Z_{21}^{(2)} x \partial_{21}(v) \right)$$

$$\begin{aligned}
&= \frac{1}{3} \sigma_1 \left(Z_{21}^{(2)} x \partial_{21} \partial_{32}(v) + Z_{21}^{(2)} x \partial_{31}(v) \right) + \frac{1}{3} Z_{21} x \partial_{31} \partial_{21}(v) - Z_{32} y \partial_{21}^{(3)}(v) \\
&= \frac{1}{3} Z_{21} x \partial_{21}^{(2)} \partial_{32}(v) + \frac{1}{2} Z_{21} x \partial_{21} \partial_{31}(v) - Z_{32} y \partial_{21}^{(3)}(v).
\end{aligned}$$

$$\begin{aligned}
&\bullet (\delta_{\mathcal{M}_2 \mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2 \mathcal{M}_1}) \left(Z_{32} y Z_{21}^{(10)} x(v) \right); \text{ where } v \in \mathcal{D}_{19} \otimes \mathcal{D}_0 \otimes \mathcal{D}_2 \\
&= Z_{21}^{(10)} x \partial_{32}(v) + \sigma_1 \left(Z_{21}^{(9)} x \partial_{31}(v) \right) - Z_{32} y \partial_{21}^{(10)}(v) = \frac{1}{9} Z_{21} x \partial_{21}^{(8)} \partial_{31}(v) - Z_{32} y \partial_{21}^{(8)}(v).
\end{aligned}$$

And

$$\begin{aligned}
&(\delta_{\mathcal{L}_2 \mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{L}_2 \mathcal{M}_1}) \left(\frac{1}{45} Z_{32} y Z_{21}^{(2)} x \partial_{21}^{(8)}(v) \right) \\
&= Z_{21}^{(2)} x \partial_{21}^{(8)} \partial_{32}(v) + \sigma_1 \left(\frac{1}{45} Z_{21}^{(2)} x \partial_{21}^{(7)} \partial_{31}(v) \right) + \frac{1}{45} Z_{21} x \partial_{31} \partial_{21}^{(8)}(v) - Z_{32} y \partial_{21}^{(2)} \partial_{21}^{(10)}(v) \\
&= \frac{1}{9} Z_{21} x \partial_{21}^{(8)} \partial_{31}(v) - Z_{32} y \partial_{21}^{(10)}(v).
\end{aligned}$$

$$\begin{aligned}
&\bullet (\delta_{\mathcal{M}_2 \mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2 \mathcal{M}_1}) \left(Z_{32} y Z_{32} y(v) \right); \text{ where } v \in \mathcal{D}_9 \otimes \mathcal{D}_{11} \otimes \mathcal{D}_1 \\
&= \sigma_1 \left(2 Z_{32}^{(2)} y(v) \right) - Z_{32} y \partial_{32}(v) = \frac{2}{2} Z_{32} y \partial_{32}(v) - Z_{32} y \partial_{32}(v) = 0.
\end{aligned}$$

$$\begin{aligned}
&\bullet (\delta_{\mathcal{M}_2 \mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2 \mathcal{M}_1}) \left(Z_{32}^{(2)} y Z_{21}^{(3)} x(v) \right); \text{ where } v \in \mathcal{D}_{12} \otimes \mathcal{D}_8 \otimes \mathcal{D}_1 \\
&= \sigma_1 \left(Z_{21}^{(3)} x \partial_{32}^{(2)}(v) + Z_{21}^{(2)} x \partial_{32} \partial_{31}(v) \right) + Z_{21} x \partial_{31}^{(2)}(v) - \sigma_1 \left(Z_{32}^{(2)} y \partial_{21}^{(3)}(v) \right) \\
&= \frac{1}{3} Z_{21} x \partial_{21}^{(2)} \partial_{32}^{(2)}(v) + \frac{1}{2} Z_{21} x \partial_{21} \partial_{32} \partial_{31}(v) + Z_{21} x \partial_{31}^{(2)}(v) - \frac{1}{2} Z_{32} y \partial_{32} \partial_{21}^{(3)}(v).
\end{aligned}$$

And

$$\begin{aligned}
&(\delta_{\mathcal{L}_2 \mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{L}_2 \mathcal{M}_1}) \left(\frac{1}{3} Z_{32} y Z_{21}^{(2)} x \partial_{31}(v) - \frac{1}{3} Z_{32} y Z_{31} z \partial_{21}^{(2)}(v) \right) \\
&= \sigma_1 \left(\frac{1}{3} Z_{21}^{(2)} x \partial_{32} \partial_{31}(v) \right) + \frac{1}{3} Z_{21} x \partial_{31} \partial_{31}(v) - \frac{1}{3} Z_{21} x \partial_{21}^{(2)} \partial_{31}(v) - \sigma_1 \left(\frac{1}{3} Z_{32}^{(2)} y \partial_{21} \partial_{21}^{(2)}(v) \right) + \frac{1}{3} Z_{21} x \partial_{32}^{(2)} \partial_{21}^{(2)}(v) + \\
&\frac{1}{3} Z_{32} y \partial_{31} \partial_{21}^{(2)}(v) = \frac{1}{3} Z_{21} x \partial_{21}^{(2)} \partial_{32}^{(2)}(v) + \frac{1}{2} Z_{21} x \partial_{21} \partial_{32} \partial_{31}(v) + Z_{21} x \partial_{31}^{(2)}(v) - \frac{1}{2} Z_{32} y \partial_{32} \partial_{21}^{(3)}(v).
\end{aligned}$$

$$\begin{aligned}
&\bullet (\delta_{\mathcal{M}_2 \mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2 \mathcal{M}_1}) \left(Z_{32}^{(2)} y Z_{21}^{(11)} x(v) \right); \text{ where } v \in \mathcal{D}_{20} \otimes \mathcal{D}_0 \otimes \mathcal{D}_1 \\
&= Z_{21}^{(11)} x \partial_{31}^{(2)}(v) + Z_{21}^{(10)} x \partial_{32} \partial_{31}(v) + \sigma_1 \left(Z_{21}^{(9)} x \partial_{31}^{(2)}(v) - Z_{32}^{(2)} y \partial_{21}^{(11)}(v) \right) = \frac{1}{9} Z_{21} x \partial_{21}^{(8)} \partial_{31}^{(2)}(v) - \frac{1}{2} Z_{32} y \partial_{32} \partial_{21}^{(11)}(v).
\end{aligned}$$

And

$$\begin{aligned}
&(\delta_{\mathcal{L}_2 \mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{L}_2 \mathcal{M}_1}) \left(\frac{1}{495} Z_{32} y Z_{21}^{(2)} x \partial_{21}^{(8)} \partial_{31}(v) - \frac{1}{11} Z_{32} y Z_{31} z \partial_{21}^{(11)}(v) \right) \\
&= Z_{21}^{(2)} x \partial_{21}^{(8)} \partial_{32} \partial_{31}(v) + \sigma_1 \left(\frac{1}{495} Z_{21}^{(2)} x \partial_{21}^{(7)} \partial_{31}^{(2)}(v) \right) + \frac{1}{495} Z_{21} x \partial_{31} \partial_{21}^{(8)} \partial_{31}(v) - \\
&\frac{1}{495} Z_{32} y \partial_{21}^{(2)} \partial_{21}^{(8)} \partial_{31}(v) - \sigma_1 \left(\frac{1}{11} Z_{32}^{(2)} y \partial_{21} \partial_{21}^{(10)}(v) \right) + \frac{1}{11} Z_{21} x \partial_{32}^{(2)} \partial_{21}^{(10)}(v) + \frac{1}{11} Z_{32} y \partial_{31} \partial_{21}^{(10)}(v) \\
&= \frac{1}{9} Z_{21} x \partial_{21}^{(8)} \partial_{31}^{(2)}(v) - \frac{1}{2} Z_{32} y \partial_{32} \partial_{21}^{(11)}(v).
\end{aligned}$$

$$\begin{aligned}
&\bullet (\delta_{\mathcal{M}_2 \mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2 \mathcal{M}_1}) \left(Z_{32}^{(2)} y Z_{32} y(v) \right); \text{ where } v \in \mathcal{D}_9 \otimes \mathcal{D}_{12} \otimes \mathcal{D}_0 \\
&= \sigma_1 \left(3 Z_{32}^{(3)} y(v) - Z_{32}^{(2)} y \partial_{32}(v) \right) = 0.
\end{aligned}$$

$$\begin{aligned}
&\bullet (\delta_{\mathcal{M}_2 \mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2 \mathcal{M}_1}) \left(Z_{32}^{(3)} y Z_{21}^{(4)} x(v) \right); \text{ where } v \in \mathcal{D}_{13} \otimes \mathcal{D}_5 \otimes \mathcal{D}_0 \\
&= \sigma_1 \left(Z_{21}^{(4)} x \partial_{32}^{(3)}(v) + Z_{21}^{(3)} x \partial_{32}^{(2)} \partial_{31}(v) + Z_{21}^{(2)} x \partial_{32} \partial_{31}^{(2)}(v) \right) + Z_{21} x \partial_{31}^{(3)}(v) - \sigma_1 \left(Z_{32}^{(3)} y \partial_{21}^{(4)}(v) \right) \\
&= \frac{1}{4} Z_{21} x \partial_{21}^{(3)} \partial_{32}^{(3)}(v) + \frac{1}{3} Z_{21} x \partial_{21}^{(2)} \partial_{32}^{(2)} \partial_{31}(v) + \frac{1}{2} Z_{21} x \partial_{21} \partial_{32} \partial_{31}^{(2)}(v) + \\
&Z_{21} x \partial_{31}^{(3)}(v) - \frac{1}{3} Z_{32} y \partial_{21}^{(4)} \partial_{32}^{(2)}(v) - \frac{1}{3} Z_{32} y \partial_{21}^{(3)} \partial_{32} \partial_{31}(v) - \frac{1}{3} Z_{32} y \partial_{21}^{(2)} \partial_{31}^{(2)}(v).
\end{aligned}$$

And

$$(\delta_{\mathcal{L}_2 \mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{L}_2 \mathcal{M}_1}) \left(\frac{1}{3} Z_{32} y Z_{21}^{(2)} x \partial_{31}^{(2)}(v) - \frac{1}{6} Z_{32} y Z_{21}^{(2)} x \partial_{21}^{(2)} \partial_{32}^{(2)}(v) - \frac{1}{3} Z_{32} y Z_{31} z \partial_{21}^{(3)} \partial_{32}(v) \right)$$

$$\begin{aligned}
 &= \sigma_1 \left(\frac{1}{3} Z_{21}^{(2)} x \partial_{32} \partial_{31}^{(2)}(v) \right) + \frac{1}{3} Z_{21} x \partial_{31} \partial_{31}^{(2)}(v) - \frac{1}{3} Z_{32} y \partial_{21}^{(2)} \partial_{31}^{(2)}(v) - \\
 &\quad \frac{1}{6} \sigma_1 \left(Z_{21}^{(2)} x \partial_{32} \partial_{21}^{(2)} \partial_{32}^{(2)}(v) + Z_{21}^{(2)} x \partial_{21} \partial_{31} \partial_{32}^{(2)}(v) \right) - \frac{1}{6} Z_{21} x \partial_{31} \partial_{21}^{(2)} \partial_{32}^{(2)}(v) + \\
 &\quad Z_{32} y \partial_{21}^{(4)} \partial_{32}^{(2)}(v) - \sigma_1 \left(\frac{1}{3} Z_{32}^{(2)} y \partial_{21} \partial_{21}^{(3)} \partial_{32}^{(2)}(v) \right) + \frac{1}{3} Z_{21} x \partial_{32}^{(2)} \partial_{21}^{(3)} \partial_{32}^{(2)}(v) + \frac{1}{3} Z_{32} y \partial_{31} \partial_{21}^{(3)} \partial_{32}^{(2)}(v) \\
 &= \frac{1}{4} Z_{21} x \partial_{21}^{(3)} \partial_{32}^{(3)}(v) + \frac{1}{3} Z_{21} x \partial_{21}^{(2)} \partial_{32}^{(2)} \partial_{31}^{(2)}(v) + \frac{1}{2} Z_{21} x \partial_{21} \partial_{32} \partial_{31}^{(2)}(v) + \\
 &\quad Z_{21} x \partial_{31}^{(3)}(v) - \frac{1}{3} Z_{32} y \partial_{21}^{(4)} \partial_{32}^{(2)}(v) - \frac{1}{3} Z_{32} y \partial_{21}^{(3)} \partial_{32} \partial_{31} - \frac{1}{3} Z_{32} y \partial_{21}^{(2)} \partial_{31}^{(2)}(v).
 \end{aligned}$$

$$\begin{aligned}
 &\bullet (\delta_{\mathcal{M}_2 \mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2 \mathcal{M}_1}) \left(Z_{32}^{(3)} y Z_{21}^{(12)} x(v) \right); \text{ where } v \in \mathcal{D}_{21} \otimes \mathcal{D}_0 \otimes \mathcal{D}_0 \\
 &= Z_{21}^{(12)} x \partial_{32}^{(3)}(v) + Z_{21}^{(11)} x \partial_{32}^{(2)} \partial_{31}^{(2)}(v) + Z_{21}^{(10)} x \partial_{32} \partial_{31}^{(2)}(v) + \sigma_1 \left(Z_{21}^{(9)} x \partial_{31}^{(3)}(v) - Z_{32}^{(3)} y \partial_{21}^{(12)}(v) \right) \\
 &= \frac{1}{9} Z_{21} x \partial_{21}^{(8)} \partial_{31}^{(3)}(v) - \frac{1}{3} Z_{32} y \partial_{21}^{(10)} \partial_{31}^{(2)}(v).
 \end{aligned}$$

And

$$\begin{aligned}
 &(\delta_{\mathcal{L}_2 \mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{L}_2 \mathcal{M}_1}) \left(\frac{1}{135} Z_{32} y Z_{21}^{(2)} x \partial_{21}^{(8)} \partial_{31}^{(2)}(v) \right) \\
 &= \frac{1}{135} Z_{21}^{(2)} x \partial_{21}^{(8)} \partial_{32} \partial_{31}^{(2)}(v) + \sigma_1 \left(\frac{1}{135} Z_{21}^{(2)} x \partial_{21}^{(7)} \partial_{31} \partial_{31}^{(2)}(v) \right) + \frac{1}{135} Z_{21} x \partial_{31} \partial_{21}^{(8)} \partial_{31}^{(2)}(v) - \frac{1}{135} Z_{32} y \partial_{21}^{(2)} \partial_{21}^{(8)} \partial_{31}^{(2)}(v) \\
 &= \frac{1}{9} Z_{21} x \partial_{21}^{(8)} \partial_{31}^{(3)}(v) - \frac{1}{3} Z_{32} y \partial_{21}^{(10)} \partial_{31}^{(2)}(v).
 \end{aligned}$$

$$\begin{aligned}
 &\bullet (\delta_{\mathcal{M}_2 \mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2 \mathcal{M}_1}) \left(Z_{32}^{(2)} y Z_{31} z(v) \right); \text{ where } v \in \mathcal{D}_{10} \otimes \mathcal{D}_{11} \otimes \mathcal{D}_0 \\
 &= \sigma_1 \left(Z_{32}^{(3)} y \partial_{21}^{(3)}(v) \right) - Z_{21} x \partial_{32}^{(3)}(v) - \sigma_1 \left(Z_{32}^{(2)} y \partial_{31}^{(3)}(v) \right) = \frac{1}{3} Z_{32} y \partial_{32}^{(2)} \partial_{21}^{(3)}(v) - Z_{21} x \partial_{32}^{(3)}(v) - \frac{1}{2} Z_{32} y \partial_{32} \partial_{31}^{(3)}(v).
 \end{aligned}$$

And

$$\begin{aligned}
 &(\delta_{\mathcal{L}_2 \mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{L}_2 \mathcal{M}_1}) \left(\frac{1}{3} Z_{32} y Z_{31} z \partial_{32}(v) \right) \\
 &= \sigma_1 \left(\frac{1}{3} Z_{32}^{(2)} y \partial_{21} \partial_{32}(v) \right) - \frac{1}{3} Z_{21} x \partial_{32}^{(2)} \partial_{32}(v) - \frac{1}{3} Z_{32} y \partial_{31} \partial_{32}(v) \\
 &= \frac{1}{3} Z_{32} y \partial_{32}^{(2)} \partial_{21}(v) - Z_{21} x \partial_{32}^{(3)}(v) - \frac{1}{2} Z_{32} y \partial_{32} \partial_{31}(v)
 \end{aligned}$$

Now by employ σ_2 we can also define:

$$\partial_3: \mathcal{L}_3 \longrightarrow \mathcal{L}_2 \quad \text{as} \quad \partial_3 = \delta_{\mathcal{L}_3 \mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3 \mathcal{M}_2}$$

Lemma (3.3):

The composition $\partial_2 \partial_3$ equal to zero.

Proof:

$$\begin{aligned}
 \partial_2 \partial_3(a) &= (\delta_{\mathcal{L}_2 \mathcal{L}_1}(a) + (\sigma_1 \circ \delta_{\mathcal{L}_2 \mathcal{M}_1})(a)) \circ (\delta_{\mathcal{L}_3 \mathcal{L}_2}(a) + (\sigma_2 \circ \delta_{\mathcal{L}_3 \mathcal{M}_2})(a)) \\
 &= (\delta_{\mathcal{L}_2 \mathcal{L}_1} \circ \delta_{\mathcal{L}_3 \mathcal{L}_2})(a) + (\delta_{\mathcal{L}_2 \mathcal{L}_1} \circ \sigma_2 \circ \delta_{\mathcal{L}_3 \mathcal{M}_2})(a) + (\sigma_1 \circ \delta_{\mathcal{L}_2 \mathcal{M}_1} \circ \sigma_2 \circ \delta_{\mathcal{L}_3 \mathcal{M}_2})(a).
 \end{aligned}$$

But $\delta_{\mathcal{L}_2 \mathcal{L}_1} \circ \sigma_2 + \sigma_1 \circ \delta_{\mathcal{L}_2 \mathcal{M}_1} \circ \sigma_2 = \delta_{\mathcal{M}_2 \mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2 \mathcal{M}_1}$ so we get:

$$\begin{aligned}
 \partial_2 \partial_3(a) &= (\delta_{\mathcal{L}_2 \mathcal{L}_1} \circ \delta_{\mathcal{L}_3 \mathcal{L}_2})(a) + (\delta_{\mathcal{M}_2 \mathcal{L}_1} \circ \delta_{\mathcal{L}_3 \mathcal{M}_2})(a) + (\sigma_1 \circ \delta_{\mathcal{L}_2 \mathcal{L}_1} \circ \delta_{\mathcal{L}_3 \mathcal{L}_2})(a) \\
 &\quad (\sigma_1 \circ \delta_{\mathcal{M}_2 \mathcal{M}_1} \circ \delta_{\mathcal{L}_2 \mathcal{M}_2})(a).
 \end{aligned}$$

By properties of the boundary map δ we get:

$$\partial_2 \partial_3 = 0$$

We need the definition of a map $\sigma_3: \mathcal{M}_3 \longrightarrow \mathcal{L}_3$ such that

$$\delta_{\mathcal{M}_3 \mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3 \mathcal{M}_2} = (\delta_{\mathcal{L}_3 \mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3 \mathcal{M}_2}) \circ \sigma_3$$

As follows:

- $Z_{21} x Z_{21} x Z_{21} x(v) \mapsto 0$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_6 \otimes \mathcal{D}_3$.
- $Z_{21}^{(2)} x Z_{21} x Z_{21} x(v) \mapsto 0$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_5 \otimes \mathcal{D}_3$.
- $Z_{21} x Z_{21}^{(2)} x Z_{21} x(v) \mapsto 0$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_5 \otimes \mathcal{D}_3$.
- $Z_{21} x Z_{21} x Z_{21}^{(2)} x(v) \mapsto 0$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_5 \otimes \mathcal{D}_3$.

(4)

- [illegible]

- [illegible]

- [illegible]

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- $Z_{32}^{(3)} \psi Z_{21}^{(7)} x Z_{21}^{(5)} x(v) \mapsto -\frac{14}{5} Z_{32} \psi Z_{31} z Z_{21} x \partial_{21}^{(9)} \partial_{31}(v)$; where $v \in \mathcal{D}_{21} \otimes \mathcal{D}_0 \otimes \mathcal{D}_0$.
- $Z_{32}^{(3)} \psi Z_{21}^{(6)} x Z_{21}^{(6)} x(v) \mapsto -\frac{21}{5} Z_{32} \psi Z_{31} z Z_{21} x \partial_{21}^{(9)} \partial_{31}(v)$; where $v \in \mathcal{D}_{21} \otimes \mathcal{D}_0 \otimes \mathcal{D}_0$.
- $Z_{32}^{(3)} \psi Z_{21}^{(10)} x Z_{21}^{(2)} x(v) \mapsto -\frac{1}{12} Z_{32} \psi Z_{31} z Z_{21} x \partial_{21}^{(9)} \partial_{31}(v)$; where $v \in \mathcal{D}_{21} \otimes \mathcal{D}_0 \otimes \mathcal{D}_0$.
- $Z_{32}^{(3)} \psi Z_{21}^{(11)} x Z_{21} x(v) \mapsto 0$; where $v \in \mathcal{D}_{21} \otimes \mathcal{D}_0 \otimes \mathcal{D}_0$.
- $Z_{32} \psi Z_{31} z Z_{21}^{(2)} x(v) \mapsto \frac{1}{3} Z_{32} \psi Z_{31} z Z_{21} x \partial_{21}(v)$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_8 \otimes \mathcal{D}_1$.
- $Z_{32} \psi Z_{31} z Z_{21}^{(3)} x(v) \mapsto \frac{1}{6} Z_{32} \psi Z_{31} z Z_{21} x \partial_{21}^{(2)}(v)$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_7 \otimes \mathcal{D}_1$.
- $Z_{32} \psi Z_{31} z Z_{21}^{(4)} x(v) \mapsto \frac{1}{10} Z_{32} \psi Z_{31} z Z_{21} x \partial_{21}^{(3)}(v)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_6 \otimes \mathcal{D}_1$.
- $Z_{32} \psi Z_{31} z Z_{21}^{(5)} x(v) \mapsto \frac{1}{15} Z_{32} \psi Z_{31} z Z_{21} x \partial_{21}^{(4)}(v)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_5 \otimes \mathcal{D}_1$.
- $Z_{32} \psi Z_{31} z Z_{21}^{(6)} x(v) \mapsto \frac{1}{21} Z_{32} \psi Z_{31} z Z_{21} x \partial_{21}^{(5)}(v)$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_4 \otimes \mathcal{D}_1$.
- $Z_{32} \psi Z_{31} z Z_{21}^{(7)} x(v) \mapsto \frac{1}{28} Z_{32} \psi Z_{31} z Z_{21} x \partial_{21}^{(6)}(v)$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_3 \otimes \mathcal{D}_1$.
- $Z_{32} \psi Z_{31} z Z_{21}^{(8)} x(v) \mapsto \frac{1}{36} Z_{32} \psi Z_{31} z Z_{21} x \partial_{21}^{(7)}(v)$; where $v \in \mathcal{D}_{18} \otimes \mathcal{D}_2 \otimes \mathcal{D}_1$.
- $Z_{32} \psi Z_{31} z Z_{21}^{(9)} x(v) \mapsto \frac{1}{45} Z_{32} \psi Z_{31} z Z_{21} x \partial_{21}^{(8)}(v)$; where $v \in \mathcal{D}_{19} \otimes \mathcal{D}_1 \otimes \mathcal{D}_1$.
- $Z_{32} \psi Z_{31} z Z_{21}^{(10)} x(v) \mapsto \frac{1}{55} Z_{32} \psi Z_{31} z Z_{21} x \partial_{21}^{(9)}(v)$; where $v \in \mathcal{D}_{20} \otimes \mathcal{D}_0 \otimes \mathcal{D}_1$.
- $Z_{32} \psi Z_{32} \psi Z_{31} z(v) \mapsto 0$; where $v \in \mathcal{D}_{10} \otimes \mathcal{D}_{11} \otimes \mathcal{D}_0$.
- $Z_{32}^{(2)} \psi Z_{31} z Z_{21}^{(2)} x(v) \mapsto \frac{1}{3} Z_{32} \psi Z_{31} z Z_{21} x \partial_{31}(v)$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_9 \otimes \mathcal{D}_0$.
- $Z_{32}^{(2)} \psi Z_{31} z Z_{21}^{(3)} x(v) \mapsto \frac{1}{6} Z_{32} \psi Z_{31} z Z_{21} x \partial_{21} \partial_{31}(v) - \frac{1}{12} Z_{32} \psi Z_{31} z Z_{21} x \partial_{21}^{(2)} \partial_{32}(v)$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_8 \otimes \mathcal{D}_0$.
- $Z_{32}^{(2)} \psi Z_{31} z Z_{21}^{(4)} x(v) \mapsto \frac{1}{9} Z_{32} \psi Z_{31} z Z_{21} x \partial_{21}^{(2)} \partial_{31}(v) - \frac{7}{90} Z_{32} \psi Z_{31} z Z_{21} x \partial_{21}^{(3)} \partial_{32}(v)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_7 \otimes \mathcal{D}_0$.
- $Z_{32}^{(2)} \psi Z_{31} z Z_{21}^{(5)} x(v) \mapsto \frac{1}{12} Z_{32} \psi Z_{31} z Z_{21} x \partial_{21}^{(3)} \partial_{31}(v) - \frac{1}{15} Z_{32} \psi Z_{31} z Z_{21} x \partial_{21}^{(4)} \partial_{32}(v)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_6 \otimes \mathcal{D}_0$.
- $Z_{32}^{(2)} \psi Z_{31} z Z_{21}^{(6)} x(v) \mapsto \frac{1}{15} Z_{32} \psi Z_{31} z Z_{21} x \partial_{21}^{(4)} \partial_{31}(v) - \frac{2}{35} Z_{32} \psi Z_{31} z Z_{21} x \partial_{21}^{(5)} \partial_{32}(v)$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_5 \otimes \mathcal{D}_0$.
- $Z_{32}^{(2)} \psi Z_{31} z Z_{21}^{(7)} x(v) \mapsto \frac{1}{18} Z_{32} \psi Z_{31} z Z_{21} x \partial_{21}^{(5)} \partial_{31}(v) - \frac{25}{504} Z_{32} \psi Z_{31} z Z_{21} x \partial_{21}^{(6)} \partial_{32}(v)$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_4 \otimes \mathcal{D}_0$.
- $Z_{32}^{(2)} \psi Z_{31} z Z_{21}^{(8)} x(v) \mapsto \frac{1}{21} Z_{32} \psi Z_{31} z Z_{21} x \partial_{21}^{(6)} \partial_{31}(v) + \frac{11}{252} Z_{32} \psi Z_{31} z Z_{21} x \partial_{21}^{(7)} \partial_{32}(v)$; where $v \in \mathcal{D}_{18} \otimes \mathcal{D}_3 \otimes \mathcal{D}_0$.
- $Z_{32}^{(2)} \psi Z_{31} z Z_{21}^{(9)} x(v) \mapsto \frac{7}{168} Z_{32} \psi Z_{31} z Z_{21} x \partial_{21}^{(7)} \partial_{31}(v) + \frac{11}{252} Z_{32} \psi Z_{31} z Z_{21} x \partial_{21}^{(8)} \partial_{32}(v)$; where $v \in \mathcal{D}_{19} \otimes \mathcal{D}_2 \otimes \mathcal{D}_0$.
- $Z_{32}^{(2)} \psi Z_{31} z Z_{21}^{(10)} x(v) \mapsto \frac{1}{27} Z_{32} \psi Z_{31} z Z_{21} x \partial_{21}^{(8)} \partial_{31}(v) + \frac{4}{165} Z_{32} \psi Z_{31} z Z_{21} x \partial_{21}^{(9)} \partial_{32}(v)$; where $v \in \mathcal{D}_{20} \otimes \mathcal{D}_1 \otimes \mathcal{D}_0$.
- $Z_{32}^{(2)} \psi Z_{31} z Z_{21}^{(11)} x(v) \mapsto \frac{1}{30} Z_{32} \psi Z_{31} z Z_{21} x \partial_{21}^{(9)} \partial_{31}(v)$; where $v \in \mathcal{D}_{21} \otimes \mathcal{D}_0 \otimes \mathcal{D}_0$.

Proposition (3.4):

The map σ_3 defined above satisfies (3.3).

Proof: We can see that for some terms:

$$\begin{aligned} & \bullet (\delta_{\mathcal{M}_3 \mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3 \mathcal{M}_2})(Z_{21} x Z_{21} x Z_{21} x(v)); \text{ where } v \in \mathcal{D}_{12} \otimes \mathcal{D}_6 \otimes \mathcal{D}_3 \\ &= \sigma_2 \left(2 Z_{21}^{(2)} x Z_{21} x(v) - 2 Z_{21} x Z_{21}^{(2)} x(v) + Z_{21} x Z_{21} x \partial_{21}(v) \right) = 0. \end{aligned}$$

$$\begin{aligned} & \bullet (\delta_{\mathcal{M}_3 \mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3 \mathcal{M}_2})(Z_{21} x Z_{21}^{(3)} x Z_{21}^{(2)} x(v)); \text{ where } v \in \mathcal{D}_{15} \otimes \mathcal{D}_3 \otimes \mathcal{D}_3 \\ &= \sigma_2 \left(4 Z_{21}^{(4)} x Z_{21}^{(2)} x(v) - 10 Z_{21} x Z_{21}^{(5)} x(v) + Z_{21} x Z_{21}^{(3)} x \partial_{21}^{(2)}(v) \right) = 0. \end{aligned}$$

$$\begin{aligned} & \bullet (\delta_{\mathcal{M}_3 \mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3 \mathcal{M}_2})(Z_{32} \psi Z_{32} \psi Z_{21}^{(3)} x(v)); \text{ where } v \in \mathcal{D}_{12} \otimes \mathcal{D}_8 \otimes \mathcal{D}_1 \\ &= \sigma_2 \left(2 Z_{32}^{(2)} \psi Z_{21}^{(3)} x(v) - Z_{32} \psi Z_{21}^{(3)} x \partial_{32}(v) \right) - Z_{32} \psi Z_{21}^{(2)} x \partial_{31}(v) + \sigma_2 \left(Z_{32} \psi Z_{32} \psi \partial_{21}^{(3)}(v) \right) \\ &= -\frac{1}{3} Z_{32} \psi Z_{21}^{(2)} x \partial_{31}(v) - \frac{2}{3} Z_{32} \psi Z_{31} z \partial_{21}^{(2)}(v) - \frac{1}{3} Z_{32} \psi Z_{21}^{(2)} x \partial_{21} \partial_{32}(v). \end{aligned}$$

And

$$\begin{aligned} & (\delta_{\mathcal{L}_3 \mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3 \mathcal{M}_2}) \left(-\frac{1}{3} Z_{32} \psi Z_{31} z Z_{21} x \partial_{21}(v) \right) \\ &= \sigma_2 \left(\frac{1}{3} Z_{21} x Z_{21} x \partial_{32}^{(2)} \partial_{21}(v) \right) - \frac{1}{3} Z_{32} \psi Z_{21}^{(2)} x \partial_{32} \partial_{21}(v) + \sigma_2 \left(\frac{1}{3} Z_{32} \psi Z_{32} \psi \partial_{21}^{(2)} \partial_{21}(v) \right) - \frac{2}{3} Z_{32} \psi Z_{31} z \partial_{21}^{(2)}(v) \end{aligned}$$

$$= -\frac{1}{3} Z_{32} \psi Z_{21}^{(2)} x \partial_{31}(v) - \frac{2}{3} Z_{32} \psi Z_{31} z \partial_{21}^{(2)}(v) - \frac{1}{3} Z_{32} \psi Z_{21}^{(2)} x \partial_{21} \partial_{32}(v).$$

$$\begin{aligned} & \bullet (\delta_{M_3 L_2} + \sigma_2 \circ \delta_{M_3 M_2}) \left(Z_{32} \psi Z_{32} \psi Z_{21}^{(11)} x(v) \right); \text{ where } v \in \mathcal{D}_{20} \otimes \mathcal{D}_0 \otimes \mathcal{D}_1 \\ &= \sigma_2 \left(2 Z_{32}^{(2)} \psi Z_{21}^{(11)} x(v) \right) - Z_{32} \psi Z_{21}^{(11)} x \partial_{32}(v) - \sigma_2 \left(Z_{32} \psi Z_{21}^{(10)} x \partial_{31}(v) + Z_{32} \psi Z_{32} \psi \partial_{21}^{(11)}(v) \right) \\ &= -\frac{1}{55} Z_{32} \psi Z_{21}^{(2)} x \partial_{21}^{(8)} \partial_{31}(v) - \frac{2}{11} Z_{32} \psi Z_{31} z \partial_{21}^{(10)}(v). \end{aligned}$$

And

$$\begin{aligned} & (\delta_{L_3 L_2} + \sigma_2 \circ \delta_{L_3 M_2}) \left(-\frac{1}{55} Z_{32} \psi Z_{31} z Z_{21} x \partial_{21}^{(9)}(v) \right) \\ &= \sigma_2 \left(\frac{1}{55} Z_{21} x Z_{21} x \partial_{32}^{(2)} \partial_{21}^{(9)}(v) \right) - \frac{1}{55} Z_{32} \psi Z_{21}^{(2)} x \partial_{32} \partial_{21}^{(9)}(v) + \sigma_2 \left(\frac{1}{55} Z_{32} \psi Z_{32} \psi \partial_{21}^{(2)} \partial_{21}^{(9)}(v) \right) - \frac{10}{55} Z_{32} \psi Z_{31} z \partial_{21}^{(10)}(v) \\ &= -\frac{1}{55} Z_{32} \psi Z_{21}^{(2)} x \partial_{21}^{(8)} \partial_{31}(v) - \frac{2}{11} Z_{32} \psi Z_{31} z \partial_{21}^{(10)}(v). \end{aligned}$$

$$\begin{aligned} & \bullet (\delta_{M_3 L_2} + \sigma_2 \circ \delta_{M_3 M_2}) \left(Z_{32}^{(2)} \psi Z_{21}^{(3)} x Z_{21} x(v) \right); \text{ where } v \in \mathcal{D}_{13} \otimes \mathcal{D}_7 \otimes \mathcal{D}_1 \\ &= \sigma_2 \left(Z_{21}^{(3)} x Z_{21} x \partial_{32}^{(2)}(v) + Z_{21}^{(2)} x Z_{21} x \partial_{32} \partial_{31}(v) + Z_{21} x Z_{21} x \partial_{31}^{(2)}(v) - 4 Z_{32}^{(2)} \psi Z_{21}^{(4)} x(v) + Z_{32}^{(2)} \psi Z_{21}^{(3)} x \partial_{21}(v) \right) = 0. \end{aligned}$$

$$\begin{aligned} & \bullet (\delta_{M_3 L_2} + \sigma_2 \circ \delta_{M_3 M_2}) \left(Z_{32} \psi Z_{32} \psi Z_{32} \psi(v) \right); \text{ where } v \in \mathcal{D}_9 \otimes \mathcal{D}_{12} \otimes \mathcal{D}_0 \\ &= \sigma_2 \left(2 Z_{32}^{(2)} \psi Z_{32} \psi(v) - 2 Z_{32} \psi Z_{32}^{(2)} \psi(v) + Z_{32} \psi Z_{32} \psi \partial_{32}(v) \right) = 0. \end{aligned}$$

$$\begin{aligned} & \bullet (\delta_{M_3 L_2} + \sigma_2 \circ \delta_{M_3 M_2}) \left(Z_{32}^{(2)} \psi Z_{32} \psi Z_{21}^{(4)} x(v) \right); \text{ where } v \in \mathcal{D}_{13} \otimes \mathcal{D}_8 \otimes \mathcal{D}_0 \\ &= \sigma_2 \left(3 Z_{32}^{(3)} \psi Z_{21}^{(4)} x(v) - Z_{32}^{(2)} \psi Z_{21}^{(4)} x \partial_{32}(v) - Z_{32}^{(2)} \psi Z_{21}^{(3)} x \partial_{31}(v) + Z_{32}^{(2)} \psi Z_{32} \psi \partial_{21}^{(4)}(v) \right) \\ &= \frac{1}{3} Z_{32} \psi Z_{21}^{(2)} x \partial_{31}^{(2)}(v) - \frac{1}{2} Z_{32} \psi Z_{21}^{(2)} x \partial_{21}^{(2)} \partial_{32}^{(2)}(v) - \frac{3}{4} Z_{32} \psi Z_{31} z \partial_{21}^{(3)} \partial_{32}(v) - \frac{1}{12} Z_{32} \psi Z_{21}^{(2)} x \partial_{21} \partial_{32} \partial_{31}(v) + \\ & \frac{1}{3} Z_{32} \psi Z_{31} z \partial_{21}^{(2)} \partial_{31}(v). \end{aligned}$$

And

$$\begin{aligned} & (\delta_{L_3 L_2} + \sigma_2 \circ \delta_{L_3 M_2}) \left(\frac{1}{6} Z_{32} \psi Z_{31} z Z_{21} x \partial_{21} \partial_{31}(v) - \frac{1}{4} Z_{32} \psi Z_{31} z Z_{21} x \partial_{21}^{(2)} \partial_{32}(v) \right) \\ &= \sigma_2 \left(-\frac{1}{6} Z_{21} x Z_{21} x \partial_{32}^{(2)} \partial_{21} \partial_{31}(v) \right) + \frac{1}{6} Z_{32} \psi Z_{21}^{(2)} x \partial_{32} \partial_{21} \partial_{31}(v) + \sigma_2 \left(\frac{1}{6} Z_{32} \psi Z_{32} \psi \partial_{21}^{(2)} \partial_{21} \partial_{31}(v) \right) + \\ & \frac{1}{6} Z_{32} \psi Z_{31} z \partial_{21} \partial_{21} \partial_{31}(v) + \\ & \sigma_2 \left(\frac{1}{4} Z_{21} x Z_{21} x \partial_{32}^{(2)} \partial_{21}^{(2)} \partial_{32}(v) \right) - \frac{1}{4} Z_{32} \psi Z_{21}^{(2)} x \partial_{32} \partial_{21}^{(2)} \partial_{32}(v) + \\ & \sigma_2 \left(\frac{1}{4} Z_{32} \psi Z_{32} \psi \partial_{21}^{(2)} \partial_{21}^{(2)} \partial_{32}(v) \right) - \frac{1}{4} Z_{32} \psi Z_{31} z \partial_{21} \partial_{21}^{(2)} \partial_{32}(v) \\ &= \frac{1}{3} Z_{32} \psi Z_{21}^{(2)} x \partial_{31}^{(2)}(v) - \frac{1}{2} Z_{32} \psi Z_{21}^{(2)} x \partial_{21}^{(2)} \partial_{32}^{(2)}(v) - \frac{3}{4} Z_{32} \psi Z_{31} z \partial_{21}^{(3)} \partial_{32}(v) - \frac{1}{12} Z_{32} \psi Z_{21}^{(2)} x \partial_{21} \partial_{32} \partial_{31}(v) + \\ & \frac{1}{3} Z_{32} \psi Z_{31} z \partial_{21}^{(2)} \partial_{31}(v). \end{aligned}$$

$$\begin{aligned} & \bullet (\delta_{M_3 L_2} + \sigma_2 \circ \delta_{M_3 M_2}) \left(Z_{32}^{(2)} \psi Z_{32} \psi Z_{21}^{(12)} x(v) \right); \text{ where } v \in \mathcal{D}_{21} \otimes \mathcal{D}_0 \otimes \mathcal{D}_0 \\ &= \sigma_2 \left(3 Z_{32}^{(3)} \psi Z_{21}^{(12)} x(v) \right) - Z_{32}^{(2)} \psi Z_{21}^{(12)} x \partial_{32}(v) - \sigma_2 \left(Z_{32}^{(2)} \psi Z_{21}^{(11)} x \partial_{31}(v) + Z_{32}^{(2)} \psi Z_{32} \psi \partial_{21}^{(12)}(v) \right) \\ &= \frac{1}{55} Z_{32} \psi Z_{21}^{(2)} x \partial_{21}^{(8)} \partial_{31}(v) + \frac{1}{11} Z_{32} \psi Z_{31} z \partial_{21}^{(10)} \partial_{31}(v). \end{aligned}$$

And

$$\begin{aligned} & (\delta_{L_3 L_2} + \sigma_2 \circ \delta_{L_3 M_2}) \left(\frac{1}{110} Z_{32} \psi Z_{31} z Z_{21} x \partial_{21}^{(9)} \partial_{31}(v) \right) \\ &= \sigma_2 \left(-\frac{1}{110} Z_{21} x Z_{21} x \partial_{32}^{(2)} \partial_{21}^{(9)} \partial_{31}(v) \right) + \frac{1}{110} Z_{32} \psi Z_{21}^{(2)} x \partial_{32} \partial_{21}^{(9)} \partial_{31}(v) - \\ & \sigma_2 \left(\frac{1}{110} Z_{32} \psi Z_{32} \psi \partial_{21}^{(2)} \partial_{21}^{(9)} \partial_{31}(v) \right) + \frac{1}{110} Z_{32} \psi Z_{31} z \partial_{21} \partial_{21}^{(9)} \partial_{31}(v) = \frac{1}{55} Z_{32} \psi Z_{21}^{(2)} x \partial_{21}^{(8)} \partial_{31}^{(2)}(v) + \\ & \frac{1}{11} Z_{32} \psi Z_{31} z \partial_{21}^{(10)} \partial_{31}(v). \end{aligned}$$

$$\begin{aligned}
& \bullet (\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}) \left(Z_{32} \psi Z_{32}^{(2)} \psi Z_{21}^{(4)} x(v) \right); \text{ where } v \in \mathcal{D}_{13} \otimes \mathcal{D}_8 \otimes \mathcal{D}_0 \\
& = \sigma_2 \left(3 Z_{32}^{(3)} \psi Z_{21}^{(4)} x(v) - Z_{32} \psi Z_{21}^{(4)} x \partial_{32}^{(2)}(v) - Z_{32} \psi Z_{21}^{(3)} x \partial_{32} \partial_{31}(v) - \right. \\
& \quad \left. Z_{32} \psi Z_{21}^{(2)} x \partial_{31}^{(2)}(v) + Z_{32} \psi Z_{32}^{(2)} \psi \partial_{21}^{(4)}(v) \right) \\
& = -\frac{2}{3} Z_{32} \psi Z_{21}^{(2)} x \partial_{21}^{(2)} \partial_{32}^{(2)}(v) - Z_{32} \psi Z_{31} \partial_{21}^{(3)} \partial_{32}(v) - \frac{1}{3} Z_{32} \psi Z_{21}^{(2)} x \partial_{21} \partial_{32} \partial_{31}(v).
\end{aligned}$$

And

$$\begin{aligned}
& (\delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2}) \left(-\frac{1}{3} Z_{32} \psi Z_{31} \partial_{21}^{(2)} \partial_{32}(v) \right) \\
& = \sigma_2 \left(\frac{1}{3} Z_{21} x Z_{21} x \partial_{32}^{(2)} \partial_{21}^{(2)} \partial_{32}(v) \right) - \frac{1}{3} Z_{32} \psi Z_{21}^{(2)} x \partial_{32} \partial_{21}^{(2)} \partial_{32}(v) + \\
& \quad \sigma_2 \left(\frac{1}{3} Z_{32} \psi Z_{32} \psi \partial_{21}^{(2)} \partial_{21}^{(2)} \partial_{32}(v) \right) - \frac{1}{3} Z_{32} \psi Z_{31} \partial_{21} \partial_{21}^{(2)} \partial_{32}(v) \\
& = -\frac{2}{3} Z_{32} \psi Z_{21}^{(2)} x \partial_{21}^{(2)} \partial_{32}^{(2)}(v) - Z_{32} \psi Z_{31} \partial_{21}^{(3)} \partial_{32}(v) - \frac{1}{3} Z_{32} \psi Z_{21}^{(2)} x \partial_{21} \partial_{32} \partial_{31}(v).
\end{aligned}$$

$$\begin{aligned}
& \bullet (\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}) \left(Z_{32} \psi Z_{32}^{(2)} \psi Z_{21}^{(12)} x(v) \right); \text{ where } v \in \mathcal{D}_{21} \otimes \mathcal{D}_0 \otimes \mathcal{D}_0 \\
& = \sigma_2 \left(3 Z_{32}^{(3)} \psi Z_{21}^{(12)} x(v) \right) - Z_{32} \psi Z_{21}^{(12)} x \partial_{32}^{(2)}(v) - Z_{32} \psi Z_{21}^{(11)} x \partial_{32} \partial_{31}(v) - \\
& \quad \sigma_2 \left(-Z_{32} \psi Z_{21}^{(10)} x \partial_{31}^{(2)}(v) + Z_{32} \psi Z_{32}^{(2)} \psi \partial_{21}^{(12)}(v) \right) = \frac{3}{135} Z_{32} \psi Z_{21}^{(2)} x \partial_{21}^{(8)} \partial_{31}^{(2)}(v) - \frac{1}{45} Z_{32} \psi Z_{21}^{(2)} x \partial_{21}^{(8)} \partial_{31}^{(2)}(v) = 0.
\end{aligned}$$

$$\begin{aligned}
& \bullet (\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}) \left(Z_{32}^{(3)} \psi Z_{21}^{(4)} x Z_{21} x(v) \right); \text{ where } v \in \mathcal{D}_{14} \otimes \mathcal{D}_7 \otimes \mathcal{D}_0 \\
& = \sigma_2 \left(Z_{21}^{(4)} x Z_{21} x \partial_{32}^{(2)}(v) + Z_{21}^{(3)} x Z_{21} x \partial_{32}^{(2)} \partial_{31}(v) + Z_{21}^{(2)} x Z_{21} x \partial_{32} \partial_{31}^{(2)}(v) + \right. \\
& \quad \left. Z_{21} x Z_{21} x \partial_{31}^{(3)}(v) - 5 Z_{32}^{(3)} \psi Z_{21}^{(5)} x(v) + Z_{32}^{(3)} \psi Z_{21}^{(4)} x \partial_{21}(v) \right) \\
& = -\frac{2}{9} Z_{32} \psi Z_{21}^{(2)} x \partial_{21} \partial_{31}^{(2)}(v) - \frac{1}{9} Z_{32} \psi Z_{21}^{(2)} x \partial_{21}^{(3)} \partial_{32}^{(2)}(v) - \frac{2}{9} Z_{32} \psi Z_{31} \partial_{21}^{(4)} \partial_{32}(v) \\
& \quad - \frac{1}{6} Z_{32} \psi Z_{21}^{(2)} x \partial_{21}^{(2)} \partial_{32} \partial_{31}(v) - \frac{1}{3} Z_{32} \psi Z_{31} \partial_{21}^{(3)} \partial_{31}(v).
\end{aligned}$$

And

$$\begin{aligned}
& (\delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2}) \left(-\frac{1}{9} Z_{32} \psi Z_{31} \partial_{21}^{(2)} \partial_{31}(v) - \frac{1}{18} Z_{32} \psi Z_{31} \partial_{21}^{(3)} \partial_{32}(v) \right) \\
& = \sigma_2 \left(\frac{1}{9} Z_{21} x Z_{21} x \partial_{32}^{(2)} \partial_{21}^{(2)} \partial_{31}(v) \right) - \frac{1}{9} Z_{32} \psi Z_{21}^{(2)} x \partial_{32} \partial_{21}^{(2)} \partial_{31}(v) + \\
& \quad \sigma_2 \left(\frac{1}{9} Z_{32} \psi Z_{32} \psi \partial_{21}^{(2)} \partial_{21}^{(2)} \partial_{31}(v) \right) - \frac{1}{9} Z_{32} \psi Z_{31} \partial_{21} \partial_{21}^{(2)} \partial_{31}(v) + \\
& \quad \sigma_2 \left(\frac{1}{18} Z_{21} x Z_{21} x \partial_{32}^{(2)} \partial_{21}^{(3)} \partial_{32}(v) \right) - \frac{1}{18} Z_{32} \psi Z_{21}^{(2)} x \partial_{32} \partial_{21}^{(3)} \partial_{32}(v) + \\
& \quad \sigma_2 \left(\frac{1}{18} Z_{32} \psi Z_{32} \psi \partial_{21}^{(2)} \partial_{21}^{(3)} \partial_{32}(v) \right) - \frac{1}{18} Z_{32} \psi Z_{31} \partial_{21} \partial_{21}^{(3)} \partial_{32}(v) \\
& = -\frac{2}{9} Z_{32} \psi Z_{21}^{(2)} x \partial_{21} \partial_{31}^{(2)}(v) - \frac{1}{9} Z_{32} \psi Z_{21}^{(2)} x \partial_{21}^{(3)} \partial_{32}^{(2)}(v) - \frac{2}{9} Z_{32} \psi Z_{31} \partial_{21}^{(4)} \partial_{32}(v) \\
& \quad - \frac{1}{6} Z_{32} \psi Z_{21}^{(2)} x \partial_{21}^{(2)} \partial_{32} \partial_{31}(v) - \frac{1}{3} Z_{32} \psi Z_{31} \partial_{21}^{(3)} \partial_{31}(v).
\end{aligned}$$

$$\begin{aligned}
& \bullet (\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}) \left(Z_{32}^{(3)} \psi Z_{21}^{(11)} x Z_{21} x(v) \right); \text{ where } v \in \mathcal{D}_{21} \otimes \mathcal{D}_0 \otimes \mathcal{D}_0 \\
& = Z_{21}^{(11)} x Z_{21} x \partial_{32}^{(3)}(v) + Z_{21}^{(10)} x Z_{21} x \partial_{32}^{(2)} \partial_{31}(v) + Z_{21}^{(9)} x Z_{21} x \partial_{32} \partial_{31}^{(2)}(v) + \\
& \quad \sigma_2 \left(Z_{21}^{(8)} x Z_{21} x \partial_{31}^{(3)}(v) - 12 Z_{32}^{(3)} \psi Z_{21}^{(10)} x(v) + Z_{32}^{(3)} \psi Z_{21}^{(11)} x \partial_{21}(v) \right) = 0.
\end{aligned}$$

$$\begin{aligned}
& \bullet (\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}) \left(Z_{32} \psi Z_{31} \partial_{21}^{(2)} x(v) \right); \text{ where } v \in \mathcal{D}_{12} \otimes \mathcal{D}_8 \otimes \mathcal{D}_1 \\
& = \sigma_2 \left(Z_{32}^{(2)} \psi Z_{21}^{(3)} x(v) - Z_{21} x Z_{21}^{(2)} x \partial_{32}^{(2)}(v) + Z_{32} \psi Z_{21}^{(3)} x \partial_{32}(v) - \right. \\
& \quad \left. Z_{32} \psi Z_{32} \psi \partial_{21}^{(3)}(v) \right) + Z_{32} \psi Z_{31} \partial_{21}^{(2)}(v) = \frac{1}{3} Z_{32} \psi Z_{21}^{(2)} x \partial_{31}(v) + \frac{2}{3} Z_{32} \psi Z_{31} \partial_{21}^{(2)}(v) + \frac{1}{3} Z_{32} \psi Z_{21}^{(2)} x \partial_{21} \partial_{32}(v).
\end{aligned}$$

And

$$(\delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2}) \left(\frac{1}{3} Z_{32} \psi Z_{31} \partial_{21} x(v) \right)$$

$$= \sigma_2 \left(-\frac{1}{3} Z_{21} x Z_{21} x \partial_{32}^{(2)} \partial_{21}(v) \right) + \frac{1}{3} Z_{32} y Z_{21}^{(2)} x \partial_{32} \partial_{21}(v) - \sigma_2 \left(\frac{1}{3} Z_{32} y Z_{32} y \partial_{21}^{(2)} \partial_{21}(v) \right) + \frac{1}{3} Z_{32} y Z_{31} z \partial_{21} \partial_{21}(v) \\ = \frac{1}{3} Z_{32} y Z_{21}^{(2)} x \partial_{31}(v) + \frac{2}{3} Z_{32} y Z_{31} z \partial_{21}^{(2)}(v) + \frac{1}{3} Z_{32} y Z_{21}^{(2)} x \partial_{21} \partial_{32}(v).$$

$$\bullet (\delta_{\mathcal{M}_3 \mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3 \mathcal{M}_2}) (Z_{32} y Z_{32} y Z_{31} z(v)); \text{ where } v \in \mathcal{D}_{10} \otimes \mathcal{D}_{11} \otimes \mathcal{D}_0 \\ = \sigma_2 \left(2 Z_{32}^{(2)} y Z_{31} z(v) - 2 Z_{32} y Z_{31} z(v) + Z_{32} y Z_{32} y \partial_{31}(v) \right) = 0.$$

$$\bullet (\delta_{\mathcal{M}_3 \mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3 \mathcal{M}_2}) (Z_{32}^{(2)} y Z_{31} z Z_{21}^{(2)} x(v)); \text{ where } v \in \mathcal{D}_{12} \otimes \mathcal{D}_6 \otimes \mathcal{D}_0 \\ = \sigma_2 \left(-Z_{21} x Z_{21}^{(2)} x \partial_{32}^{(2)}(v) + Z_{21}^{(2)} x Z_{21}^{(3)} x \partial_{32}(v) + Z_{32}^{(2)} y Z_{32} y \partial_{21}^{(3)}(v) + Z_{32} y Z_{31} z \partial_{21}^{(2)}(v) \right) \\ = \frac{1}{3} Z_{32} y Z_{21}^{(2)} x \partial_{32} \partial_{31}(v) + \frac{1}{3} Z_{32} y Z_{31} z \partial_{21} \partial_{31}(v).$$

And

$$(\delta_{\mathcal{L}_3 \mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3 \mathcal{M}_2}) \left(\frac{1}{3} Z_{32} y Z_{31} z Z_{21} x \partial_{31}(v) \right) \\ = \sigma_2 \left(-\frac{1}{3} Z_{21} x Z_{21} x \partial_{32}^{(2)} \partial_{31}(v) \right) + \frac{1}{3} Z_{32} y Z_{21}^{(2)} x \partial_{32} \partial_{31}(v) - \sigma_2 \left(\frac{1}{3} Z_{32} y Z_{32} y \partial_{21}^{(2)} \partial_{31}(v) \right) + \frac{1}{3} Z_{32} y Z_{31} z \partial_{21} \partial_{31}(v) \\ = \frac{1}{3} Z_{32} y Z_{21}^{(2)} x \partial_{32} \partial_{31}(v) + \frac{1}{3} Z_{32} y Z_{31} z \partial_{21} \partial_{31}(v).$$

$$\bullet (\delta_{\mathcal{M}_3 \mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3 \mathcal{M}_2}) (Z_{32}^{(2)} y Z_{31} z Z_{21}^{(11)} x(v)); \text{ where } v \in \mathcal{D}_{21} \otimes \mathcal{D}_0 \otimes \mathcal{D}_0 \\ = \sigma_2 \left(9 Z_{32}^{(3)} y Z_{21}^{(12)} x(v) \right) - Z_{21} x Z_{21}^{(11)} x \partial_{32}^{(3)}(v) + Z_{32}^{(2)} y Z_{21}^{(12)} x \partial_{32} - \sigma_2 \left(Z_{32}^{(2)} y Z_{32} y \partial_{21}^{(12)}(v) + Z_{32}^{(2)} y Z_{31} z \partial_{21}^{(11)}(v) \right) \\ = \frac{1}{15} Z_{32} y Z_{21}^{(2)} x \partial_{21}^{(8)} \partial_{31}^{(2)}(v) + \frac{1}{3} Z_{32} y Z_{31} z \partial_{21}^{(10)} \partial_{31}(v).$$

And

$$(\delta_{\mathcal{L}_3 \mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3 \mathcal{M}_2}) \left(\frac{1}{30} Z_{32} y Z_{31} z Z_{21} x \partial_{21}^{(9)} \partial_{31}(v) \right) \\ = \sigma_2 \left(-\frac{1}{30} Z_{21} x Z_{21} x \partial_{32}^{(2)} \partial_{21}^{(9)} \partial_{31}(v) \right) + \frac{1}{30} Z_{32} y Z_{21}^{(2)} x \partial_{32} \partial_{21}^{(9)} \partial_{31}(v) - \\ \sigma_2 \left(\frac{1}{30} Z_{32} y Z_{32} y \partial_{21}^{(2)} \partial_{21}^{(9)} \partial_{31}(v) \right) + \frac{1}{24} Z_{32} y Z_{31} z \partial_{21} \partial_{21}^{(9)} \partial_{31}(v) \\ = \frac{1}{15} Z_{32} y Z_{21}^{(2)} x \partial_{21}^{(8)} \partial_{31}^{(2)}(v) + \frac{1}{3} Z_{32} y Z_{31} z \partial_{21}^{(10)} \partial_{31}(v).$$

Eventually, we define the boundary maps in the complex:

$$0 \longrightarrow \mathcal{L}_3 \xrightarrow{\partial_3} \mathcal{L}_2 \xrightarrow{\partial_2} \mathcal{L}_1 \xrightarrow{\partial_1} \mathcal{L}_0; \quad (5)$$

where ∂_1 is the operation of indicated polarization operators, ∂_1 , ∂_2 and ∂_3 defined as follows:

- $\partial_1(Z_{21}x(v)) = \partial_{21}(v)$; where $v \in \mathcal{D}_{10} \otimes \mathcal{D}_8 \otimes \mathcal{D}_3$.
- $\partial_1(Z_{32}y(v)) = \partial_{32}(v)$; where $v \in \mathcal{D}_9 \otimes \mathcal{D}_{10} \otimes \mathcal{D}_2$.
- $\partial_2(Z_{32}yZ_{21}^{(2)}x(v)) = \frac{1}{2} Z_{21}x\partial_{21}\partial_{32}(v) + Z_{21}x\partial_{31}(v) - Z_{32}y\partial_{21}^{(2)}(v)$; where $v \in \mathcal{D}_{11} \otimes \mathcal{D}_8 \otimes \mathcal{D}_2$.
- $\partial_2(Z_{32}yZ_{31}z(v)) = \frac{1}{2} Z_{32}y\partial_{32}\partial_{21}(v) - Z_{21}x\partial_{32}^{(2)}(v) - Z_{32}y\partial_{31}(v)$; where $v \in \mathcal{D}_{10} \otimes \mathcal{D}_{10} \otimes \mathcal{D}_1$.
- $\partial_3(Z_{32}yZ_{31}zZ_{21}x(v)) = Z_{32}yZ_{21}^{(2)}x\partial_{32}(v) + Z_{32}yZ_{31}z\partial_{21}(v)$; where $v \in \mathcal{D}_{11} \otimes \mathcal{D}_9 \otimes \mathcal{D}_1$.

Theorem (3.5):

The complex (3.4) is exact and in characteristic-zero gives a resolution of $K_{(9,9,3)}(\mathcal{F})$.

Proof:

First, we prove the exactness of the complex

$$0 \longrightarrow \mathcal{L}_3 \xrightarrow{\partial_3} \mathcal{L}_2 \xrightarrow{\partial_2} \mathcal{L}_1.$$

Since one component of the map ∂_3 is a diagonalization of $\mathcal{D}_2$ into $\mathcal{D}_1 \otimes \mathcal{D}_1$ it is clear that ∂_3 is injective. To prove the exactness at $\mathcal{L}_2$.

For this, we need to show that:

If $v \in \ker(\partial_2)$ then $\exists w \in \mathcal{L}_3$ such that $\partial_3(w) = v$.

If $\partial_2(v) = 0$ then $\exists (a, b) \in \mathcal{L}_3 \oplus \mathcal{M}_3$ such that

$\delta(a, b) = (v, 0) \in \mathcal{L}_2 \oplus \mathcal{M}_2$, but

$\delta(a, b) = \delta_{\mathcal{L}_3 \mathcal{L}_2}(a) + \delta_{\mathcal{L}_3 \mathcal{M}_2}(a) + \delta_{\mathcal{M}_2 \mathcal{L}_2}(b) + \delta_{\mathcal{M}_3 \mathcal{M}_2}(b)$. So we get:

$$\delta_{\mathcal{L}_3 \mathcal{L}_2}(a) + \delta_{\mathcal{M}_3 \mathcal{L}_2}(b) = v \quad (6)$$

and

$$\delta_{\mathcal{L}_3 \mathcal{M}_2}(a) + \delta_{\mathcal{M}_3 \mathcal{M}_2}(b) = 0 \quad (7)$$

Now if $w = a + \sigma_3(b)$ we can see that $\partial_3(w) = v$ in fact

$\partial_3(a) = \delta_{\mathcal{L}_3 \mathcal{L}_2}(a) + \sigma_2 \circ \delta_{\mathcal{L}_3 \mathcal{M}_2}(a)$, and

$\partial_3(\sigma_3(b)) = \delta_{\mathcal{M}_3 \mathcal{L}_2}(b) + \sigma_2 \circ \delta_{\mathcal{M}_3 \mathcal{M}_2}(b)$, so

$$\begin{aligned} \partial_3(a + \sigma_3(b)) &= \partial_3(a) + \partial_3(\sigma_3(b)) \\ &= \delta_{\mathcal{L}_3 \mathcal{L}_2}(a) + \sigma_2 \circ \delta_{\mathcal{L}_3 \mathcal{M}_2}(a) + \delta_{\mathcal{M}_2 \mathcal{L}_2}(b) + \sigma_2 \circ \delta_{\mathcal{M}_3 \mathcal{M}_2}(b) \\ &= \delta_{\mathcal{L}_3 \mathcal{L}_2}(a) + \delta_{\mathcal{M}_3 \mathcal{L}_2}(b) + \sigma_2 \circ \left(\delta_{\mathcal{L}_3 \mathcal{M}_2}(a) + \delta_{\mathcal{M}_3 \mathcal{M}_2}(b) \right). \end{aligned}$$

Hence from (1) and (2), we get $\partial_3(w) = v$; where $w = a + \sigma_3(b)$.

This proves the exactness at $\mathcal{L}_2$.

As the same way we can prove the exactness at $\mathcal{L}_1$.

Finally, we get the complex:

$$0 \longrightarrow \mathcal{L}_3 \xrightarrow{\partial_3} \mathcal{L}_2 \xrightarrow{\partial_2} \mathcal{L}_1 \xrightarrow{\partial_1} \mathcal{L}_0 \longrightarrow \mathcal{K}_{(9,9,3)}(\mathcal{F}) \longrightarrow 0,$$

is exact.

Conflicts Of Interest

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