



Research Article

Score for the Partition (9,8,3)

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ABSTRACT

The Weyl module resolution studied by Buchsbaum where the Weyl module $\mathcal{K}_{\lambda/\mu}(\mathcal{F})$ is the image of the Weyl map $d'_{\lambda/\mu}(\mathcal{F})$ for the skew-partition λ/μ and \mathcal{F} is a free module defined on a commutative ring \mathcal{R} with identity; where λ runs over all partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_s)$. There are a number of classical formulas that express the formal character of the representation $\mathcal{L}_\lambda(\mathcal{F})$ in terms of standard symmetric polynomials. Such formulas are also valid for the more general representation modules $\{\mathcal{L}_{\lambda/\mu}(\mathcal{F})\}$ associated to skew partition λ/μ ; where $\mu \subseteq \lambda$, where the set of all irreducible polynomial representations of general linear group $GL_n(\mathcal{F})$ of degree n is described by the module $\{\mathcal{L}_\lambda(\mathcal{F})\}$

The reduction from the terms of the characteristic-free of Weyl module resolution to the terms of the Lascoux resolution found in this work for the partition (9,8,3) by using the boundary maps and prove that the sequence of the reduction terms is exact.

1. INTRODUCTION

The terms of the resolutions for all shapes called class of almost skew shapes. This characterization is largely located on the Bar complex framework, but a total characterization of the boundary map is still an open problem the researchers in [1,2] studied that in details.

The authors in [3] presented the skeleton in the resolution of skew-shapes. Especially the terms of Lascoux resolution can be recovered within the formulas approaching in [2,4]. Over and above the application of the outcomes aforesaid above, the authors in [5,6] illustrated that by employing the letter place methods and place polarization in a symmetric way.

The authors in [7] studied the corresponding of Weyl module to the partition (2,2,2), the relationship between the resolution of $\mathcal{K}_{(2,2,2)}\mathcal{F}$ in the characteristic-free module and in the Lascoux mode. By this comparison, the characteristic-free boundary maps are modified to obtain the obvious maps of the Lascoux case.

The techniques in [7] generalized by Hassan for the partitions (3,3,3), and (4,4,3) in [8,9] respectively, also authors in [10-12] studied the cases (8,7,3), (6,6,4;0,0), (7,7,4;0,0).

The terms of characteristic-free resolution of Weyl module reduction to the terms of Lascoux resolution for the partition (9,8,3) and prove that the sequence of these terms is exact in this work.

2. THE CHARACTERISTIC-FREE AND LASCoux RESOLUTION

The terms of the resolution for the partition (9,8,3)

$$Res([9,8; 0]) \otimes \mathcal{D}_3 \oplus \sum_{e \geq 0} \mathcal{Z}_{32}^{(e+1)} \psi Res([9,8 + e + 1; e + 1]) \otimes \mathcal{D}_{3-e-1} \oplus$$

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$$\sum_{e_1 \geq 0, e_2 \geq e_1} \underline{Z}_{32}^{(e_2+1)} \psi \underline{Z}_{31}^{(e_1+1)} z \text{Res}([9 + e_1 + 1, 8 + e_2 + 1; e_2 - e_1]) \otimes \mathcal{D}_{3-(e_1+e_2+2)} \tag{1}$$

So

$$\sum_{e \geq 0} \underline{Z}_{32}^{(e+1)} \psi \text{Res}([9, 8 + e + 1; e + 1]) \otimes \mathcal{D}_{3-e-1} = \underline{Z}_{32} \psi \text{Res}([9, 9; 1]) \otimes \mathcal{D}_2 \oplus \underline{Z}_{32}^{(2)} \psi \text{Res}([9, 10; 2]) \otimes \mathcal{D}_1 \oplus \underline{Z}_{32}^{(3)} \psi \text{Res}([9, 11; 3]) \otimes \mathcal{D}_0,$$

and

$$\sum_{e_1 \geq 0, e_2 \geq e_1} \underline{Z}_{32}^{(e_2+1)} \psi \underline{Z}_{31}^{(e_1+1)} z \text{Res}([9 + e_1 + 1, 8 + e_2 + 1; e_2 - e_1]) \otimes \mathcal{D}_{3-(e_1+e_2+2)} = \underline{Z}_{32} \psi \underline{Z}_{31} z \text{Res}([10, 9; 0]) \otimes \mathcal{D}_1 \oplus \underline{Z}_{32}^{(2)} \psi \underline{Z}_{31} z \text{Res}([10, 10; 1]) \otimes \mathcal{D}_0;$$

and

$$\sum_{e_1 \geq 0, e_2 \geq e_1} \underline{Z}_{32}^{(e_2+1)} \psi \underline{Z}_{31}^{(e_1+1)} z \text{Res}([9 + e_1 + 1, 9 + e_2 + 1; e_2 - e_1]) \otimes \mathcal{D}_{3-(e_1+e_2+2)} = \underline{Z}_{32} \psi \underline{Z}_{31} z \text{Res}([10, 10; 0]) \otimes \mathcal{D}_1 \oplus \underline{Z}_{32}^{(2)} \psi \underline{Z}_{31} z \text{Res}([10, 11; 1]) \otimes \mathcal{D}_0;$$

where $\underline{Z}_{32} \psi$ is the Bar complex:

$$0 \rightarrow \underline{Z}_{32} \psi \xrightarrow{\partial_\psi} \underline{Z}_{32} \rightarrow 0,$$

$\underline{Z}_{32}^{(2)} \psi$ is the Bar complex:

$$0 \rightarrow \underline{Z}_{32} \psi \underline{Z}_{32} \psi \xrightarrow{\partial_\psi} \underline{Z}_{32}^{(2)} \psi \xrightarrow{\partial_\psi} \underline{Z}_{32}^{(2)} \rightarrow 0,$$

$\underline{Z}_{32}^{(3)} \psi$ is the Bar complex:

$$0 \rightarrow \underline{Z}_{32} \psi \underline{Z}_{32} \psi \underline{Z}_{32} \psi \xrightarrow{\partial_\psi} \begin{matrix} \underline{Z}_{32}^{(2)} \psi \underline{Z}_{32} \psi \\ \oplus \\ \underline{Z}_{32} \psi \underline{Z}_{32}^{(2)} \psi \end{matrix} \xrightarrow{\partial_\psi} \underline{Z}_{32}^{(3)} \psi \xrightarrow{\partial_\psi} \underline{Z}_{32}^{(3)} \rightarrow 0,$$

and $\underline{Z}_{31} z$ is the Bar complex:

$$0 \rightarrow \underline{Z}_{31} z \xrightarrow{\partial_z} \underline{Z}_{31} \rightarrow 0;$$

where x, ψ and z stand for the separator variables, and the boundary map is $\partial_x + \partial_\psi + \partial_z$.

Let $\text{Bar}(\mathcal{M}, \mathcal{A}; \mathcal{S})$ be the free Bar module on the set $\mathcal{S} = \{x, \psi, z\}$; where \mathcal{A} is the free associative algebra generated by $\mathcal{Z}_{21}, \mathcal{Z}_{32}$, and \mathcal{Z}_{31} and their divided powers with the following relations:

$$\mathcal{Z}_{32}^{(a)} \mathcal{Z}_{31}^{(b)} = \mathcal{Z}_{31}^{(b)} \mathcal{Z}_{32}^{(a)} \quad \text{and} \quad \mathcal{Z}_{21}^{(a)} \mathcal{Z}_{31}^{(b)} = \mathcal{Z}_{31}^{(b)} \mathcal{Z}_{21}^{(a)}.$$

And the module \mathcal{M} is the direct sum of $\mathcal{D}_p \otimes \mathcal{D}_q \otimes \mathcal{D}_r$ for suitable p, q , and r with the action of $\mathcal{Z}_{21}, \mathcal{Z}_{32}$, and \mathcal{Z}_{31} and their divided powers.

The terms of the characteristic-free resolution (4.3.1); where $b, b_1, b_2, b_3, b_4, b_5, b_6, b_7, c_1, c_2 \in \mathbb{Z}^+$ are:

- In dimension zero (\mathcal{X}_0) we have $\mathcal{D}_9 \otimes \mathcal{D}_8 \otimes \mathcal{D}_3$.
- In dimension one (\mathcal{X}_1) we have the sum of the following terms:
 - $\mathcal{Z}_{21}^{(b)} x \mathcal{D}_{9+b} \otimes \mathcal{D}_{8-b} \otimes \mathcal{D}_3$; where $1 \leq b \leq 8$.
 - $\mathcal{Z}_{32}^{(b)} \psi \mathcal{D}_9 \otimes \mathcal{D}_{8+b} \otimes \mathcal{D}_{3-b}$; where $1 \leq b \leq 3$.
- In dimension two (\mathcal{X}_2) we have the sum of the following terms:
 - $\mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{8-|b|} \otimes \mathcal{D}_3$; where $2 \leq |b| = b_1 + b_2 \leq 8$.
 - $\mathcal{Z}_{32} \psi \mathcal{Z}_{21}^{(b)} x \mathcal{D}_{9+b} \otimes \mathcal{D}_{9-b} \otimes \mathcal{D}_2$; where $2 \leq b \leq 9$.
 - $\mathcal{Z}_{32}^{(2)} \psi \mathcal{Z}_{21}^{(b)} x \mathcal{D}_{9+b} \otimes \mathcal{D}_{10-b} \otimes \mathcal{D}_1$; where $3 \leq b \leq 10$.
 - $\mathcal{Z}_{32}^{(3)} \psi \mathcal{Z}_{21}^{(b)} x \mathcal{D}_{9+b} \otimes \mathcal{D}_{11-b} \otimes \mathcal{D}_0$; where $4 \leq b \leq 11$.
 - $\mathcal{Z}_{32}^{(b_1)} \psi \mathcal{Z}_{32}^{(b_2)} \psi \mathcal{D}_9 \otimes \mathcal{D}_{11+|b|} \otimes \mathcal{D}_{3-|b|}$; where $2 \leq |b| = b_1 + b_2 \leq 3$.
 - $\mathcal{Z}_{32}^{(b)} \psi \mathcal{Z}_{31} z \mathcal{D}_{10} \otimes \mathcal{D}_{10+b} \otimes \mathcal{D}_{2-b}$; where $1 \leq b \leq 2$.
- In dimension three (\mathcal{X}_3) we have the sum of the following terms:
 - $\mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{8-|b|} \otimes \mathcal{D}_3$; where $3 \leq |b| = \sum_{i=1}^3 b_i \leq 8$ and $b_1 \geq 1$.
 - $\mathcal{Z}_{32} \psi \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{9-|b|} \otimes \mathcal{D}_2$; where $3 \leq |b| = b_1 + b_2 \leq 9$ and $b_1 \geq 2$.
 - $\mathcal{Z}_{32}^{(2)} \psi \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{10-|b|} \otimes \mathcal{D}_1$; where $4 \leq |b| = b_1 + b_2 \leq 10$ and $b_1 \geq 3$.

- $Z_{32}yZ_{32}yZ_{21}^{(b)}x\mathcal{D}_{9+b}\otimes\mathcal{D}_{10-b}\otimes\mathcal{D}_1$; where $3 \leq b \leq 10$.
- $Z_{32}^{(3)}yZ_{21}^{(b_1)}xZ_{21}^{(b_2)}x\mathcal{D}_{9+|b|}\otimes\mathcal{D}_{11-|b|}\otimes\mathcal{D}_0$; where $5 \leq |b| = b_1 + b_2 \leq 12$ and $b_1 \geq 4$.
- $Z_{32}^{(c_1)}yZ_{32}^{(c_2)}yZ_{21}^{(b)}x\mathcal{D}_{9+b}\otimes\mathcal{D}_{11-b}\otimes\mathcal{D}_0$; where $c_1 + c_2 = 3$ and $4 \leq b \leq 11$.
- $Z_{32}yZ_{32}yZ_{32}y\mathcal{D}_9\otimes\mathcal{D}_{11}\otimes\mathcal{D}_0$.
- $Z_{32}yZ_{31}zZ_{21}^{(b)}x\mathcal{D}_{10+b}\otimes\mathcal{D}_{9-b}\otimes\mathcal{D}_1$; where $1 \leq b \leq 9$.
- $Z_{32}^{(2)}yZ_{31}zZ_{21}^{(b)}x\mathcal{D}_{10+b}\otimes\mathcal{D}_{10-b}\otimes\mathcal{D}_0$; where $2 \leq b \leq 10$.
- $Z_{32}yZ_{32}yZ_{31}z\mathcal{D}_{10}\otimes\mathcal{D}_{10}\otimes\mathcal{D}_0$.

◦ In dimension four (\mathcal{X}_4) we have the sum of the following terms:

- $Z_{21}^{(b_1)}xZ_{21}^{(b_2)}xZ_{21}^{(b_3)}xZ_{21}^{(b_4)}x\mathcal{D}_{9+|b|}\otimes\mathcal{D}_{8-|b|}\otimes\mathcal{D}_3$; where $4 \leq |b| = \sum_{i=1}^4 b_i \leq 8$ and $b_1 \geq 1$.
- $Z_{32}yZ_{21}^{(b_1)}xZ_{21}^{(b_2)}xZ_{21}^{(b_3)}x\mathcal{D}_{9+|b|}\otimes\mathcal{D}_{9-|b|}\otimes\mathcal{D}_2$; where $4 \leq |b| = \sum_{i=1}^3 b_i \leq 9$ and $b_1 \geq 2$.
- $Z_{32}^{(2)}yZ_{21}^{(b_1)}xZ_{21}^{(b_2)}xZ_{21}^{(b_3)}x\mathcal{D}_{9+|b|}\otimes\mathcal{D}_{10-|b|}\otimes\mathcal{D}_1$; where $5 \leq |b| = \sum_{i=1}^3 b_i \leq 10$ and $b_1 \geq 3$.
- $Z_{32}yZ_{32}yZ_{21}^{(b_1)}xZ_{21}^{(b_2)}x\mathcal{D}_{9+|b|}\otimes\mathcal{D}_{10-|b|}\otimes\mathcal{D}_1$; where $4 \leq |b| = b_1 + b_2 \leq 10$; and $b_1 \geq 3$.
- $Z_{32}^{(3)}yZ_{21}^{(b_1)}xZ_{21}^{(b_2)}xZ_{21}^{(b_3)}x\mathcal{D}_{9+|b|}\otimes\mathcal{D}_{11-|b|}\otimes\mathcal{D}_0$; where $6 \leq |b| = \sum_{i=1}^3 b_i \leq 11$ and $b_1 \geq 4$.
- $Z_{32}^{(c_1)}yZ_{32}^{(c_2)}yZ_{21}^{(b_1)}xZ_{21}^{(b_2)}x\mathcal{D}_{9+|b|}\otimes\mathcal{D}_{11-|b|}\otimes\mathcal{D}_0$; where $c_1 + c_2 = 3, 5 \leq |b| = b_1 + b_2 \leq 11$ and $b_1 \geq 4$.
- $Z_{32}yZ_{32}yZ_{32}yZ_{21}^{(b)}x\mathcal{D}_{9+b}\otimes\mathcal{D}_{11-b}\otimes\mathcal{D}_0$; where $4 \leq b \leq 11$.
- $Z_{32}yZ_{31}zZ_{21}^{(b_1)}xZ_{21}^{(b_2)}x\mathcal{D}_{10+|b|}\otimes\mathcal{D}_{9-|b|}\otimes\mathcal{D}_1$; where $2 \leq |b| = b_1 + b_2 \leq 9$ and $b_1 \geq 1$.
- $Z_{32}^{(2)}yZ_{31}zZ_{21}^{(b_1)}xZ_{21}^{(b_2)}x\mathcal{D}_{10+|b|}\otimes\mathcal{D}_{10-|b|}\otimes\mathcal{D}_0$; where $3 \leq |b| = b_1 + b_2 \leq 10$ and $b_1 \geq 2$.
- $Z_{32}yZ_{32}yZ_{31}zZ_{21}^{(b)}x\mathcal{D}_{10+b}\otimes\mathcal{D}_{10-b}\otimes\mathcal{D}_0$; where $2 \leq b \leq 10$.

◦ In dimension five (\mathcal{X}_5) we have the sum of the following terms:

- $Z_{21}^{(b_1)}xZ_{21}^{(b_2)}xZ_{21}^{(b_3)}xZ_{21}^{(b_4)}xZ_{21}^{(b_5)}x\mathcal{D}_{9+|b|}\otimes\mathcal{D}_{8-|b|}\otimes\mathcal{D}_3$; where $5 \leq |b| = \sum_{i=1}^5 b_i \leq 8$ and $b_1 \geq 1$.
- $Z_{32}yZ_{21}^{(b_1)}xZ_{21}^{(b_2)}xZ_{21}^{(b_3)}xZ_{21}^{(b_4)}x\mathcal{D}_{9+|b|}\otimes\mathcal{D}_{9-|b|}\otimes\mathcal{D}_2$; where $5 \leq |b| = \sum_{i=1}^4 b_i \leq 9$ and $b_1 \geq 2$.
- $Z_{32}^{(2)}yZ_{21}^{(b_1)}xZ_{21}^{(b_2)}xZ_{21}^{(b_3)}xZ_{21}^{(b_4)}x\mathcal{D}_{9+|b|}\otimes\mathcal{D}_{10-|b|}\otimes\mathcal{D}_1$; where $6 \leq |b| = \sum_{i=1}^4 b_i \leq 10$ and $b_1 \geq 3$.
- $Z_{32}yZ_{32}yZ_{21}^{(b_1)}xZ_{21}^{(b_2)}xZ_{21}^{(b_3)}x\mathcal{D}_{9+|b|}\otimes\mathcal{D}_{10-|b|}\otimes\mathcal{D}_1$; where $5 \leq |b| = \sum_{i=1}^3 b_i \leq 10$ and $b_1 \geq 3$.
- $Z_{32}^{(3)}yZ_{21}^{(b_1)}xZ_{21}^{(b_2)}xZ_{21}^{(b_3)}xZ_{21}^{(b_4)}x\mathcal{D}_{9+|b|}\otimes\mathcal{D}_{11-|b|}\otimes\mathcal{D}_0$; where $7 \leq |b| = \sum_{i=1}^4 b_i \leq 11$ and $b_1 \geq 4$.
- $Z_{32}^{(c_1)}yZ_{32}^{(c_2)}yZ_{21}^{(b_1)}xZ_{21}^{(b_2)}xZ_{21}^{(b_3)}x\mathcal{D}_{9+|b|}\otimes\mathcal{D}_{11-|b|}\otimes\mathcal{D}_0$; where $c_1 + c_2 = 3, 6 \leq |b| = \sum_{i=1}^3 b_i \leq 11$ and $b_1 \geq 4$.
- $Z_{32}yZ_{32}yZ_{32}yZ_{21}^{(b_1)}xZ_{21}^{(b_2)}x\mathcal{D}_{9+|b|}\otimes\mathcal{D}_{11-|b|}\otimes\mathcal{D}_0$; where $5 \leq |b| = b_1 + b_2 \leq 11$ and $b_1 \geq 4$.
- $Z_{32}yZ_{31}zZ_{21}^{(b_1)}xZ_{21}^{(b_2)}xZ_{21}^{(b_3)}x\mathcal{D}_{10+|b|}\otimes\mathcal{D}_{9-|b|}\otimes\mathcal{D}_1$; where $3 \leq |b| = \sum_{i=1}^3 b_i \leq 9$ and $b_1 \geq 1$.
- $Z_{32}^{(2)}yZ_{31}zZ_{21}^{(b_1)}xZ_{21}^{(b_2)}xZ_{21}^{(b_3)}x\mathcal{D}_{10+|b|}\otimes\mathcal{D}_{10-|b|}\otimes\mathcal{D}_0$; where $4 \leq |b| = \sum_{i=1}^3 b_i \leq 10$ and $b_1 \geq 2$.
- $Z_{32}yZ_{32}yZ_{31}zZ_{21}^{(b_1)}xZ_{21}^{(b_2)}x\mathcal{D}_{10+|b|}\otimes\mathcal{D}_{10-|b|}\otimes\mathcal{D}_0$; where $3 \leq |b| = b_1 + b_2 \leq 10$ and $b_1 \geq 2$.

◦ In dimension six (\mathcal{X}_6) we have the sum of the following terms:

- $Z_{21}^{(b_1)}xZ_{21}^{(b_2)}xZ_{21}^{(b_3)}xZ_{21}^{(b_4)}xZ_{21}^{(b_5)}xZ_{21}^{(b_6)}x\mathcal{D}_{9+|b|}\otimes\mathcal{D}_{8-|b|}\otimes\mathcal{D}_3$; where $6 \leq |b| = \sum_{i=1}^6 b_i \leq 8$ and $b_1 \geq 1$.
- $Z_{32}yZ_{21}^{(b_1)}xZ_{21}^{(b_2)}xZ_{21}^{(b_3)}xZ_{21}^{(b_4)}xZ_{21}^{(b_5)}x\mathcal{D}_{9+|b|}\otimes\mathcal{D}_{9-|b|}\otimes\mathcal{D}_2$; where $6 \leq |b| = \sum_{i=1}^5 b_i \leq 9$ and $b_1 \geq 2$.
- $Z_{32}^{(2)}yZ_{21}^{(b_1)}xZ_{21}^{(b_2)}xZ_{21}^{(b_3)}xZ_{21}^{(b_4)}xZ_{21}^{(b_5)}x\mathcal{D}_{9+|b|}\otimes\mathcal{D}_{10-|b|}\otimes\mathcal{D}_1$; where $7 \leq |b| = \sum_{i=1}^5 b_i \leq 10$ and $b_1 \geq 3$.
- $Z_{32}yZ_{32}yZ_{21}^{(b_1)}xZ_{21}^{(b_2)}xZ_{21}^{(b_3)}xZ_{21}^{(b_4)}x\mathcal{D}_{9+|b|}\otimes\mathcal{D}_{10-|b|}\otimes\mathcal{D}_1$; where $6 \leq |b| = \sum_{i=1}^4 b_i \leq 10$ and $b_1 \geq 3$.
- $Z_{32}^{(3)}yZ_{21}^{(b_1)}xZ_{21}^{(b_2)}xZ_{21}^{(b_3)}xZ_{21}^{(b_4)}xZ_{21}^{(b_5)}x\mathcal{D}_{9+|b|}\otimes\mathcal{D}_{11-|b|}\otimes\mathcal{D}_0$; where $8 \leq |b| = \sum_{i=1}^5 b_i \leq 11$ and $b_1 \geq 4$.
- $Z_{32}^{(c_1)}yZ_{32}^{(c_2)}yZ_{21}^{(b_1)}xZ_{21}^{(b_2)}xZ_{21}^{(b_3)}xZ_{21}^{(b_4)}x\mathcal{D}_{9+|b|}\otimes\mathcal{D}_{11-|b|}\otimes\mathcal{D}_0$; where $c_1 + c_2 = 3, 7 \leq |b| = \sum_{i=1}^4 b_i \leq 11$ and $b_1 \geq 4$.
- $Z_{32}yZ_{32}yZ_{32}yZ_{21}^{(b_1)}xZ_{21}^{(b_2)}xZ_{21}^{(b_3)}x\mathcal{D}_{9+|b|}\otimes\mathcal{D}_{11-|b|}\otimes\mathcal{D}_0$; where $6 \leq |b| = \sum_{i=1}^3 b_i \leq 11$ and $b_1 \geq 4$.
- $Z_{32}yZ_{31}zZ_{21}^{(b_1)}xZ_{21}^{(b_2)}xZ_{21}^{(b_3)}xZ_{21}^{(b_4)}x\mathcal{D}_{10+|b|}\otimes\mathcal{D}_{9-|b|}\otimes\mathcal{D}_1$; where $4 \leq |b| = \sum_{i=1}^4 b_i \leq 9$ and $b_1 \geq 1$.
- $Z_{32}^{(2)}yZ_{31}zZ_{21}^{(b_1)}xZ_{21}^{(b_2)}xZ_{21}^{(b_3)}xZ_{21}^{(b_4)}x\mathcal{D}_{10+|b|}\otimes\mathcal{D}_{10-|b|}\otimes\mathcal{D}_0$; where $5 \leq |b| = \sum_{i=1}^4 b_i \leq 10$ and $b_1 \geq 2$.
- $Z_{32}yZ_{32}yZ_{31}zZ_{21}^{(b_1)}xZ_{21}^{(b_2)}xZ_{21}^{(b_3)}x\mathcal{D}_{10+|b|}\otimes\mathcal{D}_{10-|b|}\otimes\mathcal{D}_0$; where $4 \leq |b| = \sum_{i=1}^3 b_i \leq 10$ and $b_1 \geq 2$.

◦ In dimension seven (\mathcal{X}_7) we have the sum of the following terms:

- $Z_{32}^{(c_1)} y Z_{32}^{(c_2)} y Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x Z_{21}^{(b_3)} x Z_{21}^{(b_4)} x Z_{21}^{(b_5)} x Z_{21}^{(b_6)} x Z_{21}^{(b_7)} x Z_{21}^{(b_8)} x D_{9+|b|} \otimes D_{11-|b|} \otimes D_0$; where $c_1 + c_2 = 3, 11 \leq |b| = \sum_{i=1}^8 b_i \leq 12$ and $b_1 \geq 4$.
- $Z_{32} y Z_{31} z Z_{21} x Z_{21} x Z_{21} x Z_{21} x Z_{21} x Z_{21} x Z_{21} x D_{19} \otimes D_0 \otimes D_1$.
- $Z_{32} y Z_{32} y Z_{31} z Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x Z_{21}^{(b_3)} x Z_{21}^{(b_4)} x Z_{21}^{(b_5)} x Z_{21}^{(b_6)} x Z_{21}^{(b_7)} x D_{10+|b|} \otimes D_{10-|b|} \otimes D_0$; where $8 \leq |b| = \sum_{i=1}^7 b_i \leq 10$ and $b_1 \geq 2$.
- $Z_{32}^{(2)} y Z_{31} z Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x Z_{21}^{(b_3)} x Z_{21}^{(b_4)} x Z_{21}^{(b_5)} x Z_{21}^{(b_6)} x Z_{21}^{(b_7)} x Z_{21}^{(b_8)} x D_{10+|b|} \otimes D_{10-|b|} \otimes D_0$; where $8 \leq |b| = \sum_{i=1}^8 b_i \leq 10$ and $b_1 \geq 2$.

◦ In dimension eleven (\mathcal{X}_{11}) we have the sum of the following terms:

- $Z_{32} y Z_{32} y Z_{32} y Z_{21}^{(4)} x Z_{21} x Z_{21} x Z_{21} x Z_{21} x Z_{21} x Z_{21} x D_{20} \otimes D_0 \otimes D_0$.
- $Z_{32} y Z_{32} y Z_{31} z Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x Z_{21}^{(b_3)} x Z_{21}^{(b_4)} x Z_{21}^{(b_5)} x Z_{21}^{(b_6)} x Z_{21}^{(b_7)} x Z_{21}^{(b_8)} x D_{10+|b|} \otimes D_{10-|b|} \otimes D_0$; where $9 \leq |b| = \sum_{i=1}^7 b_i \leq 10$ and $b_1 \geq 2$.

Finally, in dimension twelve (\mathcal{X}_{12}) we have:

- $Z_{32} y Z_{32} y Z_{31} z Z_{21}^{(2)} x Z_{21} x Z_{21} x Z_{21} x Z_{21} x Z_{21} x Z_{21} x Z_{21} x D_{20} \otimes D_0 \otimes D_0$.

The terms of the Lascoux complex are obtained by the determinantal expansion of the Jacobi-Trudi matrix of the partition [1]. The positions of the terms of the complex are determined by the length of the permutation to which they correspond, [4].

In the case of the partition (9,8,3) we get the following matrix:

$$\begin{bmatrix} D_9\mathcal{F} & D_7\mathcal{F} & D_1\mathcal{F} \\ D_{10}\mathcal{F} & D_8\mathcal{F} & D_2\mathcal{F} \\ D_{11}\mathcal{F} & D_9\mathcal{F} & D_3\mathcal{F} \end{bmatrix}$$

Then the Lascoux complex has the correspondence between its terms as pursues:

- $D_9\mathcal{F} \otimes D_8\mathcal{F} \otimes D_3\mathcal{F} \leftrightarrow \text{identity}$.
- $D_{10}\mathcal{F} \otimes D_7\mathcal{F} \otimes D_3\mathcal{F} \leftrightarrow (12)$.
- $D_9\mathcal{F} \otimes D_9\mathcal{F} \otimes D_2\mathcal{F} \leftrightarrow (23)$.
- $D_{10}\mathcal{F} \otimes D_9\mathcal{F} \otimes D_1\mathcal{F} \leftrightarrow (123)$.
- $D_{11}\mathcal{F} \otimes D_7\mathcal{F} \otimes D_2\mathcal{F} \leftrightarrow (132)$.
- $D_{11}\mathcal{F} \otimes D_8\mathcal{F} \otimes D_1\mathcal{F} \leftrightarrow (13)$

Thus the resolution of Lascoux in the case of the partition (9,8,3) has the formulation:

$$D_{11}\mathcal{F} \otimes D_8\mathcal{F} \otimes D_1\mathcal{F} \longrightarrow \begin{matrix} D_{11}\mathcal{F} \otimes D_7\mathcal{F} \otimes D_2\mathcal{F} \\ \oplus \\ D_{10}\mathcal{F} \otimes D_9\mathcal{F} \otimes D_1\mathcal{F} \end{matrix} \longrightarrow \begin{matrix} D_{10}\mathcal{F} \otimes D_7\mathcal{F} \otimes D_3\mathcal{F} \\ \oplus \\ D_9\mathcal{F} \otimes D_9\mathcal{F} \otimes D_2\mathcal{F} \end{matrix} \longrightarrow D_9\mathcal{F} \otimes D_8\mathcal{F} \otimes D_3\mathcal{F}$$

3. THE SCORE

As in [4], we exhibit the terms of the complex (2.1) as:

- $\mathcal{X}_0 = \mathcal{L}_0 = \mathcal{M}_0$,
- $\mathcal{X}_1 = \mathcal{L}_1 \oplus \mathcal{M}_1$,
- $\mathcal{X}_2 = \mathcal{L}_2 \oplus \mathcal{M}_2$,
- $\mathcal{X}_3 = \mathcal{L}_3 \oplus \mathcal{M}_3$,
- $\mathcal{X}_j = \mathcal{M}_j$; for $j = 4, 5, \dots, 12$,

where \mathcal{L}_e are the sum of the Lascoux terms and \mathcal{M}_e are the sum of the others.

Now, we define the map $\sigma_1: \mathcal{M}_1 \longrightarrow \mathcal{L}_1$ such that

$$\delta_{\mathcal{L}_1\mathcal{L}_0} \circ \sigma_1 = \delta_{\mathcal{M}_1\mathcal{M}_0} \tag{2}$$

As follows:

- $Z_{21}^{(2)} x(v) \mapsto \frac{1}{2} Z_{21} x \partial_{21}(v)$; where $v \in D_{11} \otimes D_6 \otimes D_3$.
- $Z_{21}^{(3)} x(v) \mapsto \frac{1}{3} Z_{21} x \partial_{21}^{(2)}(v)$; where $v \in D_{12} \otimes D_5 \otimes D_3$.

- $Z_{21}^{(4)}x(v) \mapsto \frac{1}{4} Z_{21}x\partial_{21}^{(3)}(v)$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_4 \otimes \mathcal{D}_3$.
- $Z_{21}^{(5)}x(v) \mapsto \frac{1}{5} Z_{21}x\partial_{21}^{(4)}(v)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_3 \otimes \mathcal{D}_3$.
- $Z_{21}^{(6)}x(v) \mapsto \frac{1}{6} Z_{21}x\partial_{21}^{(5)}(v)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_2 \otimes \mathcal{D}_3$.
- $Z_{21}^{(7)}x(v) \mapsto \frac{1}{7} Z_{21}x\partial_{21}^{(6)}(v)$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_1 \otimes \mathcal{D}_3$.
- $Z_{21}^{(8)}x(v) \mapsto \frac{1}{8} Z_{21}x\partial_{21}^{(7)}(v)$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$.
- $Z_{32}^{(2)}y(v) \mapsto \frac{1}{2} Z_{32}y\partial_{32}(v)$; where $v \in \mathcal{D}_9 \otimes \mathcal{D}_{10} \otimes \mathcal{D}_1$.
- $Z_{32}^{(3)}y(v) \mapsto \frac{1}{3} Z_{32}y\partial_{32}^{(2)}(v)$; where $v \in \mathcal{D}_9 \otimes \mathcal{D}_{11} \otimes \mathcal{D}_0$.

It is clear that σ_1 satisfies (3.1), then we can define:

$$\partial_1: \mathcal{L}_1 \longrightarrow \mathcal{L}_0 \text{ as } \partial_1 = \delta_{\mathcal{L}_1\mathcal{L}_0}$$

At this point, we are in a position to define:

$$\partial_2: \mathcal{L}_2 \longrightarrow \mathcal{L}_1 \text{ by } \partial_2 = \delta_{\mathcal{L}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{L}_2\mathcal{M}_1}$$

Lemma (3.1):

The composition $\partial_1\partial_2$ equal to zero.

Proof:

$$\begin{aligned} \partial_1\partial_2(a) &= \delta_{\mathcal{L}_1\mathcal{L}_0} \circ (\delta_{\mathcal{L}_2\mathcal{L}_1}(a) + (\sigma_1 \circ \delta_{\mathcal{L}_2\mathcal{M}_1})(a)) \\ &= \delta_{\mathcal{L}_1\mathcal{L}_0} \circ \delta_{\mathcal{L}_2\mathcal{L}_1}(a) + \delta_{\mathcal{L}_1\mathcal{L}_0} \circ (\sigma_1 \circ \delta_{\mathcal{L}_2\mathcal{M}_1})(a). \end{aligned}$$

But $\delta_{\mathcal{L}_1\mathcal{L}_0} \circ \sigma_1 = \delta_{\mathcal{M}_1\mathcal{M}_0}$ then we get:

$$\partial_1\partial_2(a) = \delta_{\mathcal{L}_1\mathcal{L}_0} \circ \delta_{\mathcal{L}_2\mathcal{L}_1}(a) + \delta_{\mathcal{M}_1\mathcal{M}_0} \circ \delta_{\mathcal{L}_2\mathcal{M}_1}(a).$$

By properties of the boundary map δ we get $\partial_1\partial_2 = 0$

We need to define the map $\sigma_2: \mathcal{M}_2 \longrightarrow \mathcal{L}_2$ such that

$$\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1} = (\delta_{\mathcal{L}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{L}_2\mathcal{M}_1}) \circ \sigma_2 \tag{3}$$

As follows:

- $Z_{21}xZ_{21}x(v) \mapsto 0$; where $v \in \mathcal{D}_{11} \otimes \mathcal{D}_6 \otimes \mathcal{D}_3$.
- $Z_{21}^{(2)}xZ_{21}x(v) \mapsto 0$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_5 \otimes \mathcal{D}_3$.
- $Z_{21}xZ_{21}^{(2)}x(v) \mapsto 0$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_5 \otimes \mathcal{D}_3$.
- $Z_{21}^{(3)}xZ_{21}x(v) \mapsto 0$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_4 \otimes \mathcal{D}_3$.
- $Z_{21}xZ_{21}^{(3)}x(v) \mapsto 0$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_4 \otimes \mathcal{D}_3$.
- $Z_{21}^{(2)}xZ_{21}^{(2)}x(v) \mapsto 0$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_4 \otimes \mathcal{D}_3$.
- $Z_{21}^{(4)}xZ_{21}x(v) \mapsto 0$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_3 \otimes \mathcal{D}_3$.
- $Z_{21}xZ_{21}^{(4)}x(v) \mapsto 0$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_3 \otimes \mathcal{D}_3$.
- $Z_{21}^{(3)}xZ_{21}^{(2)}x(v) \mapsto 0$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_3 \otimes \mathcal{D}_3$.
- $Z_{21}^{(2)}xZ_{21}^{(3)}x(v) \mapsto 0$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_3 \otimes \mathcal{D}_3$.
- $Z_{21}^{(5)}xZ_{21}x(v) \mapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_2 \otimes \mathcal{D}_3$.
- $Z_{21}xZ_{21}^{(5)}x(v) \mapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_2 \otimes \mathcal{D}_3$.
- $Z_{21}^{(3)}xZ_{21}^{(3)}x(v) \mapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_2 \otimes \mathcal{D}_3$.
- $Z_{21}^{(4)}xZ_{21}^{(2)}x(v) \mapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_2 \otimes \mathcal{D}_3$.
- $Z_{21}^{(2)}xZ_{21}^{(4)}x(v) \mapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_2 \otimes \mathcal{D}_3$.
- $Z_{21}^{(6)}xZ_{21}x(v) \mapsto 0$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_1 \otimes \mathcal{D}_3$.
- $Z_{21}xZ_{21}^{(6)}x(v) \mapsto 0$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_1 \otimes \mathcal{D}_3$.
- $Z_{21}^{(5)}xZ_{21}^{(2)}x(v) \mapsto 0$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_1 \otimes \mathcal{D}_3$.
- $Z_{21}^{(2)}xZ_{21}^{(5)}x(v) \mapsto 0$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_1 \otimes \mathcal{D}_3$.
- $Z_{21}^{(4)}xZ_{21}^{(3)}x(v) \mapsto 0$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_1 \otimes \mathcal{D}_3$.
- $Z_{21}^{(3)}xZ_{21}^{(4)}x(v) \mapsto 0$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_1 \otimes \mathcal{D}_3$.

- $Z_{21}^{(7)} x Z_{21} x(v) \mapsto 0$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$.
- $Z_{21} x Z_{21}^{(7)} x(v) \mapsto 0$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$.
- $Z_{21}^{(2)} x Z_{21}^{(6)} x(v) \mapsto 0$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$.
- $Z_{21}^{(6)} x Z_{21}^{(2)} x(v) \mapsto 0$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$.
- $Z_{21}^{(4)} x Z_{21}^{(4)} x(v) \mapsto 0$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$.
- $Z_{21}^{(3)} x Z_{21}^{(5)} x(v) \mapsto 0$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$.
- $Z_{21}^{(5)} x Z_{21}^{(3)} x(v) \mapsto 0$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$.
- $Z_{32} y Z_{21}^{(3)} x(v) \mapsto \frac{1}{3} Z_{32} y Z_{21}^{(2)} x \partial_{21}(v)$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_6 \otimes \mathcal{D}_2$.
- $Z_{32} y Z_{21}^{(4)} x(v) \mapsto \frac{1}{6} Z_{32} y Z_{21}^{(2)} x \partial_{21}^{(2)}(v)$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_5 \otimes \mathcal{D}_2$.
- $Z_{32} y Z_{21}^{(5)} x(v) \mapsto \frac{1}{10} Z_{32} y Z_{21}^{(2)} x \partial_{21}^{(3)}(v)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_4 \otimes \mathcal{D}_2$.
- $Z_{32} y Z_{21}^{(6)} x(v) \mapsto \frac{1}{15} Z_{32} y Z_{21}^{(2)} x \partial_{21}^{(4)}(v)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_3 \otimes \mathcal{D}_2$.
- $Z_{32} y Z_{21}^{(7)} x(v) \mapsto \frac{1}{21} Z_{32} y Z_{21}^{(2)} x \partial_{21}^{(5)}(v)$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_2 \otimes \mathcal{D}_2$.
- $Z_{32} y Z_{21}^{(8)} x(v) \mapsto \frac{1}{28} Z_{32} y Z_{21}^{(2)} x \partial_{21}^{(6)}(v)$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_1 \otimes \mathcal{D}_2$.
- $Z_{32} y Z_{21}^{(9)} x(v) \mapsto \frac{1}{36} Z_{32} y Z_{21}^{(2)} x \partial_{21}^{(7)}(v)$; where $v \in \mathcal{D}_{18} \otimes \mathcal{D}_0 \otimes \mathcal{D}_2$.
- $Z_{32} y Z_{32} y(v) \mapsto 0$; where $v \in \mathcal{D}_9 \otimes \mathcal{D}_{10} \otimes \mathcal{D}_1$.
- $Z_{32}^{(2)} y Z_{21}^{(3)} x(v) \mapsto \frac{1}{3} (Z_{32} y Z_{21}^{(2)} x \partial_{31}(v) - Z_{32} y Z_{31} z \partial_{21}^{(2)}(v))$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_7 \otimes \mathcal{D}_1$.
- $Z_{32}^{(2)} y Z_{21}^{(4)} x(v) \mapsto \frac{1}{12} Z_{32} y Z_{21}^{(2)} x \partial_{21} \partial_{31}(v) - \frac{1}{4} Z_{32} y Z_{31} z \partial_{21}^{(3)}(v)$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_6 \otimes \mathcal{D}_1$.
- $Z_{32}^{(2)} y Z_{21}^{(5)} x(v) \mapsto \frac{1}{30} Z_{32} y Z_{21}^{(2)} x \partial_{21}^{(2)} \partial_{31}(v) - \frac{1}{5} Z_{32} y Z_{31} z \partial_{21}^{(4)}(v)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_5 \otimes \mathcal{D}_1$.
- $Z_{32}^{(2)} y Z_{21}^{(6)} x(v) \mapsto \frac{1}{60} Z_{32} y Z_{21}^{(2)} x \partial_{21}^{(3)} \partial_{31}(v) - \frac{1}{6} Z_{32} y Z_{31} z \partial_{21}^{(5)}(v)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_4 \otimes \mathcal{D}_1$.
- $Z_{32}^{(2)} y Z_{21}^{(7)} x(v) \mapsto \frac{1}{105} Z_{32} y Z_{21}^{(2)} x \partial_{21}^{(4)} \partial_{31}(v) - \frac{1}{7} Z_{32} y Z_{31} z \partial_{21}^{(6)}(v)$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_3 \otimes \mathcal{D}_1$.
- $Z_{32}^{(2)} y Z_{21}^{(8)} x(v) \mapsto \frac{1}{168} Z_{32} y Z_{21}^{(2)} x \partial_{21}^{(5)} \partial_{31}(v) - \frac{1}{8} Z_{32} y Z_{31} z \partial_{21}^{(7)}(v)$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_2 \otimes \mathcal{D}_1$.
- $Z_{32}^{(2)} y Z_{21}^{(9)} x(v) \mapsto \frac{1}{252} Z_{32} y Z_{21}^{(2)} x \partial_{21}^{(6)} \partial_{31}(v) - \frac{1}{9} Z_{32} y Z_{31} z \partial_{21}^{(8)}(v)$; where $v \in \mathcal{D}_{18} \otimes \mathcal{D}_1 \otimes \mathcal{D}_1$.
- $Z_{32}^{(2)} y Z_{21}^{(10)} x(v) \mapsto \frac{1}{360} Z_{32} y Z_{21}^{(2)} x \partial_{21}^{(7)} \partial_{31}(v) - \frac{1}{10} Z_{32} y Z_{31} z \partial_{21}^{(9)}(v)$; where $v \in \mathcal{D}_{19} \otimes \mathcal{D}_0 \otimes \mathcal{D}_1$.
- $Z_{32}^{(2)} y Z_{32} y(v) \mapsto 0$; where $v \in \mathcal{D}_9 \otimes \mathcal{D}_{11} \otimes \mathcal{D}_0$.
- $Z_{32} y Z_{32}^{(2)} y(v) \mapsto 0$; where $v \in \mathcal{D}_9 \otimes \mathcal{D}_{11} \otimes \mathcal{D}_0$.
- $Z_{32}^{(3)} y Z_{21}^{(4)} x(v) \mapsto \frac{1}{3} Z_{32} y Z_{21}^{(2)} x \partial_{31}^{(2)}(v) - \frac{1}{6} Z_{32} y Z_{21}^{(2)} x \partial_{21}^{(2)} \partial_{32}^{(2)}(v) - \frac{1}{3} Z_{32} y Z_{31} z \partial_{21}^{(3)} \partial_{32}(v)$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_7 \otimes \mathcal{D}_0$.
- $Z_{32}^{(3)} y Z_{21}^{(5)} x(v) \mapsto \frac{1}{9} Z_{32} y Z_{21}^{(2)} x \partial_{21} \partial_{31}^{(2)}(v) - \frac{7}{90} Z_{32} y Z_{21}^{(2)} x \partial_{21}^{(3)} \partial_{32}^{(2)}(v) - \frac{2}{9} Z_{32} y Z_{31} z \partial_{21}^{(4)} \partial_{32}(v)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_6 \otimes \mathcal{D}_0$.
- $Z_{32}^{(3)} y Z_{21}^{(6)} x(v) \mapsto \frac{1}{18} Z_{32} y Z_{21}^{(2)} x \partial_{21}^{(2)} \partial_{31}^{(2)}(v) - \frac{2}{45} Z_{32} y Z_{21}^{(2)} x \partial_{21}^{(4)} \partial_{32}^{(2)}(v) - \frac{1}{6} Z_{32} y Z_{31} z \partial_{21}^{(5)} \partial_{32}(v)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_5 \otimes \mathcal{D}_0$.
- $Z_{32}^{(3)} y Z_{21}^{(7)} x(v) \mapsto \frac{1}{30} Z_{32} y Z_{21}^{(2)} x \partial_{21}^{(3)} \partial_{31}^{(2)}(v) - \frac{1}{35} Z_{32} y Z_{21}^{(2)} x \partial_{21}^{(5)} \partial_{32}^{(2)}(v) - \frac{2}{15} Z_{32} y Z_{31} z \partial_{21}^{(6)} \partial_{32}(v)$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_4 \otimes \mathcal{D}_0$.
- $Z_{32}^{(3)} y Z_{21}^{(8)} x(v) \mapsto \frac{1}{45} Z_{32} y Z_{21}^{(2)} x \partial_{21}^{(4)} \partial_{31}^{(2)}(v) - \frac{5}{252} Z_{32} y Z_{21}^{(2)} x \partial_{21}^{(6)} \partial_{32}^{(2)}(v) - \frac{1}{9} Z_{32} y Z_{31} z \partial_{21}^{(7)} \partial_{32}(v)$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_3 \otimes \mathcal{D}_0$.
- $Z_{32}^{(3)} y Z_{21}^{(9)} x(v) \mapsto \frac{1}{63} Z_{32} y Z_{21}^{(2)} x \partial_{21}^{(5)} \partial_{31}^{(2)}(v) + \frac{1}{84} Z_{32} y Z_{21}^{(2)} x \partial_{21}^{(6)} \partial_{32} \partial_{31}(v)$; where $v \in \mathcal{D}_{18} \otimes \mathcal{D}_2 \otimes \mathcal{D}_0$.
- $Z_{32}^{(3)} y Z_{21}^{(10)} x(v) \mapsto \frac{1}{84} Z_{32} y Z_{21}^{(2)} x \partial_{21}^{(6)} \partial_{31}^{(2)}(v)$; where $v \in \mathcal{D}_{19} \otimes \mathcal{D}_1 \otimes \mathcal{D}_0$.
- $Z_{32}^{(3)} y Z_{21}^{(11)} x(v) \mapsto \frac{1}{108} Z_{32} y Z_{21}^{(2)} x \partial_{21}^{(7)} \partial_{31}^{(2)}(v)$; where $v \in \mathcal{D}_{20} \otimes \mathcal{D}_0 \otimes \mathcal{D}_0$.
- $Z_{32}^{(2)} y Z_{31} z(v) \mapsto \frac{1}{3} Z_{32} y Z_{31} z \partial_{32}(v)$; where $v \in \mathcal{D}_{10} \otimes \mathcal{D}_{10} \otimes \mathcal{D}_0$.

Proposition (3.2):

The map σ_2 defined above satisfies (3.2).

Proof: We can see that for some terms:

- $(\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1})(Z_{21}xZ_{21}x(v))$; where $v \in \mathcal{D}_{11} \otimes \mathcal{D}_6 \otimes \mathcal{D}_3$
 $= \sigma_1(2Z_{21}^{(2)}x(v) - Z_{21}x\partial_{21}(v)) = \frac{2}{2}Z_{21}x\partial_{21}(v) - Z_{21}x\partial_{21}(v) = 0.$

- $(\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1})(Z_{21}^{(4)}xZ_{21}x(v))$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_3 \otimes \mathcal{D}_3$
 $= \sigma_1(5Z_{21}^{(5)}x(v) - Z_{21}^{(4)}x\partial_{21}(v)) = \frac{5}{5}Z_{21}x\partial_{21}^{(4)}(v) - \frac{1}{4}Z_{21}x\partial_{21}^{(3)}\partial_{21}(v) = 0.$

- $(\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1})(Z_{32}yZ_{21}^{(3)}x(v))$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_6 \otimes \mathcal{D}_2$
 $= \sigma_1(Z_{21}^{(3)}x\partial_{32}(v) + Z_{21}^{(2)}x\partial_{31}(v)) - Z_{32}y\partial_{21}^{(3)}(v) = \frac{1}{3}Z_{21}x\partial_{21}^{(2)}\partial_{32}(v) + \frac{1}{2}Z_{21}x\partial_{21}\partial_{31}(v) - Z_{32}y\partial_{21}^{(3)}(v).$

And

$$(\delta_{\mathcal{L}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{L}_2\mathcal{M}_1})\left(\frac{1}{3}Z_{32}yZ_{21}^{(2)}x\partial_{21}(v)\right)$$

$$= \frac{1}{3}\sigma_1\left(Z_{21}^{(2)}x\partial_{21}\partial_{32}(v) + Z_{21}^{(2)}x\partial_{31}(v)\right) + \frac{1}{3}Z_{21}x\partial_{31}\partial_{21}(v) - Z_{32}y\partial_{21}^{(3)}(v)$$

$$= \frac{1}{3}Z_{21}x\partial_{21}^{(2)}\partial_{32}(v) + \frac{1}{2}Z_{21}x\partial_{21}\partial_{31}(v) - Z_{32}y\partial_{21}^{(3)}(v).$$

- $(\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1})(Z_{32}yZ_{32}y(v))$; where $v \in \mathcal{D}_9 \otimes \mathcal{D}_{10} \otimes \mathcal{D}_1$
 $= \sigma_1(2Z_{32}^{(2)}y(v) - Z_{32}y\partial_{32}(v)) = \frac{2}{2}Z_{32}y\partial_{32}(v) - Z_{32}y\partial_{32}(v) = 0.$

- $(\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1})(Z_{32}^{(2)}yZ_{21}^{(3)}x(v))$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_7 \otimes \mathcal{D}_1$
 $= \sigma_1(Z_{21}^{(3)}x\partial_{32}^{(2)}(v) + Z_{21}^{(2)}x\partial_{32}\partial_{31}(v)) + Z_{21}x\partial_{31}^{(2)}(v) - \sigma_1(Z_{32}^{(2)}y\partial_{21}^{(3)}(v))$
 $= \frac{1}{3}Z_{21}x\partial_{21}^{(2)}\partial_{32}^{(2)}(v) + \frac{1}{2}Z_{21}x\partial_{21}\partial_{32}\partial_{31}(v) + Z_{21}x\partial_{31}^{(2)}(v) - \frac{1}{2}Z_{32}y\partial_{32}\partial_{21}^{(3)}(v).$

And

$$(\delta_{\mathcal{L}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{L}_2\mathcal{M}_1})\left(\frac{1}{3}Z_{32}yZ_{21}^{(2)}x\partial_{31}(v) - \frac{1}{3}Z_{32}yZ_{31}z\partial_{21}^{(2)}(v)\right)$$

$$= \sigma_1\left(\frac{1}{3}Z_{21}^{(2)}x\partial_{32}\partial_{31}(v)\right) + \frac{1}{3}Z_{21}x\partial_{31}\partial_{31}(v) - \frac{1}{3}Z_{21}x\partial_{21}^{(2)}\partial_{31}(v) - \sigma_1\left(\frac{1}{3}Z_{32}^{(2)}y\partial_{21}\partial_{21}^{(2)}(v)\right) + \frac{1}{3}Z_{21}x\partial_{32}^{(2)}\partial_{21}^{(2)}(v) +$$

$$\frac{1}{3}Z_{32}y\partial_{31}\partial_{21}^{(2)}(v) = \frac{1}{3}Z_{21}x\partial_{21}^{(2)}\partial_{32}^{(2)}(v) + \frac{1}{2}Z_{21}x\partial_{21}\partial_{32}\partial_{31}(v) + Z_{21}x\partial_{31}^{(2)}(v) - \frac{1}{2}Z_{32}y\partial_{32}\partial_{21}^{(3)}(v).$$

- $(\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1})(Z_{32}^{(2)}yZ_{21}^{(10)}x(v))$; where $v \in \mathcal{D}_{19} \otimes \mathcal{D}_0 \otimes \mathcal{D}_1$
 $= Z_{21}^{(10)}x\partial_{32}^{(2)}(v) + \sigma_1(Z_{21}^{(8)}x\partial_{31}^{(2)}(v) - Z_{32}^{(2)}y\partial_{21}^{(10)}(v)) = \frac{1}{8}Z_{21}x\partial_{21}^{(7)}\partial_{31}^{(2)}(v) - \frac{1}{2}Z_{32}y\partial_{32}\partial_{21}^{(10)}(v).$

And

$$(\delta_{\mathcal{L}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{L}_2\mathcal{M}_1})\left(\frac{1}{360}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(7)}\partial_{31}(v) - \frac{1}{10}Z_{32}yZ_{31}z\partial_{21}^{(9)}(v)\right) = \frac{1}{360}\sigma_1\left(Z_{21}^{(2)}x\partial_{32}\partial_{21}^{(7)}\partial_{31}(v) +$$

$$2Z_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}^{(2)}(v)\right) + \frac{1}{360}Z_{21}x\partial_{31}\partial_{21}^{(7)}\partial_{31}(v) -$$

$$\frac{36}{360}Z_{32}y\partial_{21}^{(9)}\partial_{31}(v) - \sigma_1\left(\frac{1}{10}Z_{32}^{(2)}y\partial_{21}\partial_{21}^{(9)}(v)\right) + \frac{1}{10}Z_{21}x\partial_{32}^{(2)}\partial_{21}^{(9)}(v) + \frac{1}{10}Z_{32}y\partial_{31}\partial_{21}^{(9)}(v)$$

$$= \frac{1}{8}Z_{21}x\partial_{21}^{(7)}\partial_{31}^{(2)}(v) - \frac{1}{2}Z_{32}y\partial_{32}\partial_{21}^{(10)}(v).$$

- $(\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1})(Z_{32}^{(2)}yZ_{32}y(v))$; where $v \in \mathcal{D}_9 \otimes \mathcal{D}_{11} \otimes \mathcal{D}_0$
 $= \sigma_1\left(3Z_{32}^{(3)}y(v) - Z_{32}^{(2)}y\partial_{32}(v)\right) = \frac{3}{3}Z_{32}y\partial_{32}^{(2)}(v) - \frac{2}{2}Z_{32}y\partial_{32}^{(2)}(v) = 0.$

- $(\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1})(Z_{32}^{(3)}yZ_{21}^{(4)}x(v))$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_7 \otimes \mathcal{D}_0$
 $= \sigma_1\left(Z_{21}^{(4)}x\partial_{32}^{(3)}(v) + Z_{21}^{(3)}x\partial_{32}^{(2)}\partial_{31}(v) + Z_{21}^{(2)}x\partial_{32}\partial_{31}^{(2)}(v)\right) + Z_{21}x\partial_{31}^{(3)}(v) - \sigma_1\left(Z_{32}^{(3)}y\partial_{21}^{(4)}(v)\right)$
 $= \frac{1}{4}Z_{21}x\partial_{21}^{(3)}\partial_{32}^{(3)}(v) + \frac{1}{3}Z_{21}x\partial_{21}^{(2)}\partial_{32}^{(2)}\partial_{31}(v) + \frac{1}{2}Z_{21}x\partial_{21}\partial_{32}\partial_{31}^{(2)}(v) +$
 $Z_{21}x\partial_{31}^{(3)}(v) - \frac{1}{3}Z_{32}y\partial_{21}^{(4)}\partial_{32}^{(2)}(v) - \frac{1}{3}Z_{32}y\partial_{21}^{(3)}\partial_{32}\partial_{31} - \frac{1}{3}Z_{32}y\partial_{21}^{(2)}\partial_{31}^{(2)}(v).$

And

$$\begin{aligned}
 & (\delta_{L_2L_1} + \sigma_1 \circ \delta_{L_2M_1}) \left(\frac{1}{3} Z_{32} \psi Z_{21}^{(2)} x \partial_{31}^{(2)}(v) - \frac{1}{6} Z_{32} \psi Z_{21}^{(2)} x \partial_{21}^{(2)} \partial_{32}^{(2)}(v) - \frac{1}{3} Z_{32} \psi Z_{31} z \partial_{21}^{(3)} \partial_{32}(v) \right) \\
 &= \sigma_1 \left(\frac{1}{3} Z_{21}^{(2)} x \partial_{32} \partial_{31}^{(2)}(v) \right) + \frac{1}{3} Z_{21} x \partial_{31} \partial_{31}^{(2)}(v) - \frac{1}{3} Z_{32} \psi \partial_{21}^{(2)} \partial_{31}^{(2)}(v) - \\
 & \quad \frac{1}{6} \sigma_1 \left(Z_{21}^{(2)} x \partial_{32} \partial_{21}^{(2)} \partial_{32}^{(2)}(v) + Z_{21}^{(2)} x \partial_{21} \partial_{31} \partial_{32}^{(2)}(v) \right) - \frac{1}{6} Z_{21} x \partial_{31} \partial_{21}^{(2)} \partial_{32}^{(2)}(v) + \\
 & \quad Z_{32} \psi \partial_{21}^{(4)} \partial_{32}^{(2)}(v) - \sigma_1 \left(\frac{1}{3} Z_{32}^{(2)} \psi \partial_{21} \partial_{21}^{(3)} \partial_{32}(v) \right) + \frac{1}{3} Z_{21} x \partial_{32}^{(2)} \partial_{21}^{(3)} \partial_{32}(v) + \frac{1}{3} Z_{32} \psi \partial_{31} \partial_{21}^{(3)} \partial_{32}(v) \\
 &= \frac{1}{4} Z_{21} x \partial_{21}^{(3)} \partial_{32}^{(3)}(v) + \frac{1}{3} Z_{21} x \partial_{21}^{(2)} \partial_{32}^{(2)} \partial_{31}(v) + \frac{1}{2} Z_{21} x \partial_{21} \partial_{32} \partial_{31}^{(2)}(v) + \\
 & \quad Z_{21} x \partial_{31}^{(3)}(v) - \frac{1}{3} Z_{32} \psi \partial_{21}^{(4)} \partial_{32}^{(2)}(v) - \frac{1}{3} Z_{32} \psi \partial_{21}^{(3)} \partial_{32} \partial_{31} - \frac{1}{3} Z_{32} \psi \partial_{21}^{(2)} \partial_{31}^{(2)}(v).
 \end{aligned}$$

- $(\delta_{M_2L_1} + \sigma_1 \circ \delta_{M_2M_1}) (Z_{32}^{(3)} \psi Z_{21}^{(11)} x(v))$; where $v \in \mathcal{D}_{20} \otimes \mathcal{D}_0 \otimes \mathcal{D}_0$

$$\begin{aligned}
 &= Z_{21}^{(11)} x \partial_{32}^{(3)}(v) + Z_{21}^{(10)} x \partial_{32}^{(2)} \partial_{31}(v) + \sigma_1 (Z_{21}^{(9)} x \partial_{32} \partial_{31}^{(2)}(v) + Z_{21}^{(8)} x \partial_{31}^{(3)}(v) - Z_{32}^{(3)} \psi \partial_{21}^{(11)}(v)) \\
 &= \frac{1}{8} Z_{21} x \partial_{21}^{(7)} \partial_{31}^{(3)}(v) - \frac{1}{3} Z_{32} \psi \partial_{21}^{(9)} \partial_{31}^{(2)}(v).
 \end{aligned}$$

And

$$\begin{aligned}
 & (\delta_{L_2L_1} + \sigma_1 \circ \delta_{L_2M_1}) \left(\frac{1}{108} Z_{32} \psi Z_{21}^{(2)} x \partial_{21}^{(7)} \partial_{31}^{(2)}(v) \right) \\
 &= \sigma_1 \left(\frac{1}{108} Z_{21}^{(2)} x \partial_{32} \partial_{21}^{(7)} \partial_{31}^{(2)}(v) \right) + \frac{1}{108} Z_{21} x \partial_{31} \partial_{21}^{(7)} \partial_{31}^{(2)}(v) - \frac{1}{108} Z_{32} \psi \partial_{21}^{(2)} \partial_{21}^{(7)} \partial_{31}^{(2)}(v) = \frac{1}{8} Z_{21} x \partial_{21}^{(7)} \partial_{31}^{(3)}(v) - \\
 & \quad \frac{1}{3} Z_{32} \psi \partial_{21}^{(9)} \partial_{31}^{(2)}(v).
 \end{aligned}$$

- $(\delta_{M_2L_1} + \sigma_1 \circ \delta_{M_2M_1}) (Z_{32}^{(2)} \psi Z_{31} z(v))$; where $v \in \mathcal{D}_{10} \otimes \mathcal{D}_{10} \otimes \mathcal{D}_0$

$$\begin{aligned}
 &= \sigma_1 (Z_{32}^{(3)} \psi \partial_{21}(v)) - Z_{21} x \partial_{32}^{(3)}(v) - \sigma_1 (Z_{32}^{(2)} \psi \partial_{31}(v)) = \frac{1}{3} Z_{32} \psi \partial_{32}^{(2)} \partial_{21}(v) - Z_{21} x \partial_{32}^{(3)}(v) - \frac{1}{2} Z_{32} \psi \partial_{32} \partial_{31}(v).
 \end{aligned}$$

And

$$\begin{aligned}
 & (\delta_{L_2L_1} + \sigma_1 \circ \delta_{L_2M_1}) \left(\frac{1}{3} Z_{32} \psi Z_{31} z \partial_{32}(v) \right) \\
 &= \sigma_1 \left(\frac{1}{3} Z_{32}^{(2)} \psi \partial_{21} \partial_{32}(v) \right) - \frac{1}{3} Z_{21} x \partial_{32}^{(2)} \partial_{32}(v) - \frac{1}{3} Z_{32} \psi \partial_{31} \partial_{32}(v) \\
 &= \frac{1}{3} Z_{32} \psi \partial_{32}^{(2)} \partial_{21}(v) - Z_{21} x \partial_{32}^{(3)}(v) - \frac{1}{2} Z_{32} \psi \partial_{32} \partial_{31}(v).
 \end{aligned}$$

Now by employ σ_2 we can also define

$$\partial_3: \mathcal{L}_3 \longrightarrow \mathcal{L}_2 \quad \text{as} \quad \partial_3 = \delta_{L_3L_2} + \sigma_2 \circ \delta_{L_3M_2}$$

Lemma (3.3):

The composition $\partial_2 \partial_3$ equal to zero.

Proof:

$$\begin{aligned}
 \partial_2 \partial_3(a) &= (\delta_{L_2L_1}(a) + (\sigma_1 \circ \delta_{L_2M_1})(a)) \circ (\delta_{L_3L_2}(a) + (\sigma_2 \circ \delta_{L_3M_2})(a)) \\
 &= (\delta_{L_2L_1} \circ \delta_{L_3L_2})(a) + (\delta_{L_2L_1} \circ \sigma_2 \circ \delta_{L_3M_2})(a) + (\sigma_1 \circ \delta_{L_2M_1} \circ \sigma_2 \circ \delta_{L_3M_2})(a).
 \end{aligned}$$

But $\delta_{L_2L_1} \circ \sigma_2 \circ \delta_{L_3M_2} + \sigma_1 \circ \delta_{L_2M_1} \circ \sigma_2 \circ \delta_{L_3M_2} = \delta_{M_2L_1} + \sigma_1 \circ \delta_{M_2M_1}$ so we get:

$$\partial_2 \partial_3(a) = (\delta_{L_2L_1} \circ \delta_{L_3L_2})(a) + (\delta_{M_2L_1} \circ \delta_{L_3M_2})(a) + (\sigma_1 \circ \delta_{L_2L_1} \circ \delta_{L_3L_2})(a) + (\sigma_1 \circ \delta_{M_2M_1} \circ \delta_{L_3M_2})(a).$$

By properties of the boundary map δ we get:

$$\partial_2 \partial_3 = 0$$

We need the definition of a map $\sigma_3: \mathcal{M}_3 \longrightarrow \mathcal{L}_3$ such that

$$\delta_{M_3L_2} + \sigma_2 \circ \delta_{M_3M_2} = (\delta_{L_3L_2} + \sigma_2 \circ \delta_{L_3M_2}) \circ \sigma_3 \tag{3}$$

As follows:

- $Z_{21} x Z_{21} x Z_{21} x(v) \mapsto 0$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_5 \otimes \mathcal{D}_3$.
- $Z_{21}^{(2)} x Z_{21} x Z_{21} x(v) \mapsto 0$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_4 \otimes \mathcal{D}_3$.

- $Z_{32}\psi Z_{31}zZ_{21}^{(6)}x(v) \mapsto \frac{1}{21}Z_{32}\psi Z_{31}zZ_{21}x\partial_{21}^{(5)}(v)$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_3 \otimes \mathcal{D}_1$.
- $Z_{32}\psi Z_{31}zZ_{21}^{(7)}x(v) \mapsto \frac{1}{28}Z_{32}\psi Z_{31}zZ_{21}x\partial_{21}^{(6)}(v)$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_2 \otimes \mathcal{D}_1$.
- $Z_{32}\psi Z_{31}zZ_{21}^{(8)}x(v) \mapsto \frac{1}{36}Z_{32}\psi Z_{31}zZ_{21}x\partial_{21}^{(7)}(v)$; where $v \in \mathcal{D}_{18} \otimes \mathcal{D}_1 \otimes \mathcal{D}_1$.
- $Z_{32}\psi Z_{31}zZ_{21}^{(9)}x(v) \mapsto \frac{1}{45}Z_{32}\psi Z_{31}zZ_{21}x\partial_{21}^{(8)}(v)$; where $v \in \mathcal{D}_{19} \otimes \mathcal{D}_0 \otimes \mathcal{D}_1$.
- $Z_{32}\psi Z_{31}zZ_{21}^{(10)}x(v) \mapsto 0$; where $v \in \mathcal{D}_{19} \otimes \mathcal{D}_0 \otimes \mathcal{D}_1$.
- $Z_{32}\psi Z_{32}\psi Z_{31}z(v) \mapsto 0$; where $v \in \mathcal{D}_{10} \otimes \mathcal{D}_{10} \otimes \mathcal{D}_0$.
- $Z_{32}^{(2)}\psi Z_{31}zZ_{21}^{(2)}x(v) \mapsto \frac{1}{3}Z_{32}\psi Z_{31}zZ_{21}x\partial_{31}(v)$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_8 \otimes \mathcal{D}_0$.
- $Z_{32}^{(2)}\psi Z_{31}zZ_{21}^{(3)}x(v) \mapsto \frac{1}{6}Z_{32}\psi Z_{31}zZ_{21}x\partial_{21}\partial_{31}(v) - \frac{1}{12}Z_{32}\psi Z_{31}zZ_{21}x\partial_{21}^{(2)}\partial_{32}(v)$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_7 \otimes \mathcal{D}_0$.
- $Z_{32}^{(2)}\psi Z_{31}zZ_{21}^{(4)}x(v) \mapsto \frac{1}{9}Z_{32}\psi Z_{31}zZ_{21}x\partial_{21}^{(2)}\partial_{31}(v) - \frac{7}{90}Z_{32}\psi Z_{31}zZ_{21}x\partial_{21}^{(3)}\partial_{32}(v)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_6 \otimes \mathcal{D}_0$.
- $Z_{32}^{(2)}\psi Z_{31}zZ_{21}^{(5)}x(v) \mapsto \frac{1}{12}Z_{32}\psi Z_{31}zZ_{21}x\partial_{21}^{(3)}\partial_{31}(v) - \frac{1}{15}Z_{32}\psi Z_{31}zZ_{21}x\partial_{21}^{(4)}\partial_{32}(v)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_5 \otimes \mathcal{D}_0$.
- $Z_{32}^{(2)}\psi Z_{31}zZ_{21}^{(6)}x(v) \mapsto \frac{1}{15}Z_{32}\psi Z_{31}zZ_{21}x\partial_{21}^{(4)}\partial_{31}(v) - \frac{2}{35}Z_{32}\psi Z_{31}zZ_{21}x\partial_{21}^{(5)}\partial_{32}(v)$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_4 \otimes \mathcal{D}_0$.
- $Z_{32}^{(2)}\psi Z_{31}zZ_{21}^{(7)}x(v) \mapsto \frac{1}{18}Z_{32}\psi Z_{31}zZ_{21}x\partial_{21}^{(5)}\partial_{31}(v) - \frac{25}{504}Z_{32}\psi Z_{31}zZ_{21}x\partial_{21}^{(6)}\partial_{32}(v)$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_3 \otimes \mathcal{D}_0$.
- $Z_{32}^{(2)}\psi Z_{31}zZ_{21}^{(8)}x(v) \mapsto \frac{1}{21}Z_{32}\psi Z_{31}zZ_{21}x\partial_{21}^{(6)}\partial_{31}(v) + \frac{1}{36}Z_{32}\psi Z_{31}zZ_{21}x\partial_{21}^{(7)}\partial_{32}(v)$; where $v \in \mathcal{D}_{18} \otimes \mathcal{D}_2 \otimes \mathcal{D}_0$.
- $Z_{32}^{(2)}\psi Z_{31}zZ_{21}^{(9)}x(v) \mapsto \frac{1}{24}Z_{32}\psi Z_{31}zZ_{21}x\partial_{21}^{(7)}\partial_{31}(v) - \frac{7}{180}Z_{32}\psi Z_{31}zZ_{21}x\partial_{21}^{(8)}\partial_{32}(v)$; where $v \in \mathcal{D}_{19} \otimes \mathcal{D}_1 \otimes \mathcal{D}_0$.
- $Z_{32}^{(2)}\psi Z_{31}zZ_{21}^{(10)}x(v) \mapsto \frac{1}{27}Z_{32}\psi Z_{31}zZ_{21}x\partial_{21}^{(8)}\partial_{31}(v)$; where $v \in \mathcal{D}_{20} \otimes \mathcal{D}_0 \otimes \mathcal{D}_0$.

Proposition (3.4):

The map σ_3 defined above satisfies (3.3).

Proof:

We can see that for some terms:

• $(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2})(Z_{21}xZ_{21}xZ_{21}x(v))$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_5 \otimes \mathcal{D}_3$
 $= \sigma_2 (2Z_{21}^{(2)}xZ_{21}x(v) - 2Z_{21}xZ_{21}^{(2)}x(v) + Z_{21}xZ_{21}x\partial_{21}(v)) = 0$.

• $(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2})(Z_{32}\psi Z_{21}^{(2)}xZ_{21}^{(2)}x(v))$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_5 \otimes \mathcal{D}_2$
 $= \sigma_2 (Z_{21}^{(2)}xZ_{21}^{(2)}x\partial_{32}(v) + Z_{21}xZ_{21}^{(2)}x\partial_{31}(v) - 6Z_{32}\psi Z_{21}^{(3)}x(v) + Z_{32}\psi Z_{21}^{(2)}x\partial_{21}(v))$
 $= -\frac{6}{6}Z_{32}\psi Z_{21}^{(2)}x\partial_{21}(v) + Z_{32}\psi Z_{21}^{(2)}x\partial_{21}(v) = 0$.

• $(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2})(Z_{32}\psi Z_{32}\psi Z_{21}^{(3)}x(v))$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_7 \otimes \mathcal{D}_1$
 $= \sigma_2 (2Z_{32}^{(2)}\psi Z_{21}^{(3)}x(v) - Z_{32}\psi Z_{21}^{(3)}x\partial_{32}(v)) - Z_{32}\psi Z_{21}^{(2)}x\partial_{31}(v) + \sigma_2 (Z_{32}\psi Z_{32}\psi\partial_{21}^{(3)}(v))$
 $= -\frac{1}{3}Z_{32}\psi Z_{21}^{(2)}x\partial_{31}(v) - \frac{2}{3}Z_{32}\psi Z_{31}z\partial_{21}^{(2)}(v) - \frac{1}{3}Z_{32}\psi Z_{21}^{(2)}x\partial_{21}\partial_{32}(v)$.

And

$(\delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2})\left(-\frac{1}{3}Z_{32}\psi Z_{31}zZ_{21}x\partial_{21}(v)\right)$
 $= \sigma_2 \left(\frac{1}{3}Z_{21}xZ_{21}x\partial_{32}^{(2)}\partial_{21}(v)\right) - \frac{1}{3}Z_{32}\psi Z_{21}^{(2)}x\partial_{32}\partial_{21}(v) + \sigma_2 \left(\frac{1}{3}Z_{32}\psi Z_{32}\psi\partial_{21}^{(2)}\partial_{21}(v)\right) - \frac{2}{3}Z_{32}\psi Z_{31}z\partial_{21}^{(2)}(v)$
 $= -\frac{1}{3}Z_{32}\psi Z_{21}^{(2)}x\partial_{31}(v) - \frac{2}{3}Z_{32}\psi Z_{31}z\partial_{21}^{(2)}(v) - \frac{1}{3}Z_{32}\psi Z_{21}^{(2)}x\partial_{21}\partial_{32}(v)$.

• $(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2})(Z_{32}\psi Z_{32}\psi Z_{21}^{(10)}x(v))$; where $v \in \mathcal{D}_{19} \otimes \mathcal{D}_0 \otimes \mathcal{D}_1$
 $= \sigma_2 (2Z_{32}^{(2)}\psi Z_{21}^{(10)}x(v) - Z_{32}\psi Z_{21}^{(10)}x\partial_{32}(v) - \sigma_2 (Z_{32}\psi Z_{21}^{(9)}x\partial_{31}(v) + Z_{32}\psi Z_{32}\psi\partial_{21}^{(10)}(v))$
 $= -\frac{1}{45}Z_{32}\psi Z_{21}^{(2)}x\partial_{21}^{(7)}\partial_{31}(v) - \frac{1}{5}Z_{32}\psi Z_{31}z\partial_{21}^{(9)}(v)$.

And

$(\delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2})\left(-\frac{1}{45}Z_{32}\psi Z_{31}zZ_{21}x\partial_{21}^{(8)}(v)\right)$

$$= \sigma_2 \left(\frac{1}{45} Z_{21} x Z_{21} x \partial_{32}^{(2)} \partial_{21}^{(8)}(v) \right) - \frac{1}{45} Z_{32} y Z_{21} x \partial_{32} \partial_{21}^{(8)}(v) + \sigma_2 \left(\frac{1}{45} Z_{32} y Z_{32} y \partial_{21}^{(2)} \partial_{21}^{(8)}(v) \right) - \frac{9}{45} Z_{32} y Z_{31} z \partial_{21}^{(9)}(v) \\ = -\frac{1}{45} Z_{32} y Z_{21} x \partial_{21}^{(7)} \partial_{31}(v) - \frac{1}{5} Z_{32} y Z_{31} z \partial_{21}^{(9)}(v).$$

- $(\delta_{M_3 L_2} + \sigma_2 \circ \delta_{M_3 M_2}) (Z_{32}^{(2)} y Z_{21}^{(3)} x Z_{21} x(v));$ where $v \in D_{13} \otimes D_6 \otimes D_1$
 $= \sigma_2 (Z_{21}^{(3)} x Z_{21} x \partial_{32}^{(2)}(v) + Z_{21}^{(2)} x Z_{21} x \partial_{32} \partial_{31}(v) + Z_{21} x Z_{21} x \partial_{31}^{(2)}(v) - 4 Z_{32}^{(2)} y Z_{21}^{(4)} x(v) + Z_{32}^{(2)} y Z_{21}^{(3)} x \partial_{21}(v))$
 $= -\frac{1}{3} Z_{32} y Z_{21}^{(2)} x \partial_{21} \partial_{31}(v) + Z_{32} y Z_{31} z \partial_{21}^{(3)}(v) + \frac{1}{3} Z_{32} y Z_{21}^{(2)} x \partial_{31} \partial_{21}(v) - \frac{3}{3} Z_{32} y Z_{31} z \partial_{21}^{(3)}(v) = 0.$

- $(\delta_{M_3 L_2} + \sigma_2 \circ \delta_{M_3 M_2}) (Z_{32}^{(2)} y Z_{21}^{(4)} x Z_{21} x(v));$ where $v \in D_{14} \otimes D_5 \otimes D_1$
 $= \sigma_2 (Z_{21}^{(4)} x Z_{21} x \partial_{32}^{(2)}(v) + Z_{21}^{(3)} x Z_{21} x \partial_{32} \partial_{31}(v) + Z_{21}^{(2)} x Z_{21} x \partial_{31}^{(2)}(v) - 5 Z_{32}^{(2)} y Z_{21}^{(5)} x(v) + Z_{32}^{(2)} y Z_{21}^{(4)} x \partial_{21}(v)) = 0.$

- $(\delta_{M_3 L_2} + \sigma_2 \circ \delta_{M_3 M_2}) (Z_{32} y Z_{32} y Z_{32} y(v));$ where $v \in D_9 \otimes D_{11} \otimes D_0$
 $= \sigma_2 (2 Z_{32}^{(2)} y Z_{32} y(v) - 2 Z_{32} y Z_{32}^{(2)} y(v) + Z_{32} y Z_{32} y \partial_{32}(v)) = 0.$

- $(\delta_{M_3 L_2} + \sigma_2 \circ \delta_{M_3 M_2}) (Z_{32}^{(2)} y Z_{32} y Z_{21}^{(4)} x(v));$ where $v \in D_{13} \otimes D_7 \otimes D_0$
 $= \sigma_2 (3 Z_{32}^{(3)} y Z_{21}^{(4)} x(v) - Z_{32}^{(2)} y Z_{21}^{(4)} x \partial_{32}(v) - Z_{32}^{(2)} y Z_{21}^{(3)} x \partial_{31}(v) + Z_{32}^{(2)} y Z_{32} y \partial_{21}^{(4)}(v))$
 $= \frac{1}{3} Z_{32} y Z_{21}^{(2)} x \partial_{31}^{(2)}(v) - \frac{1}{2} Z_{32} y Z_{21}^{(2)} x \partial_{21}^{(2)} \partial_{32}^{(2)}(v) - \frac{3}{4} Z_{32} y Z_{31} z \partial_{21}^{(3)} \partial_{32}(v) - \frac{1}{12} Z_{32} y Z_{21}^{(2)} x \partial_{21} \partial_{32} \partial_{31}(v) +$
 $\frac{1}{3} Z_{32} y Z_{31} z \partial_{21}^{(2)} \partial_{31}(v).$

And

$$(\delta_{L_3 L_2} + \sigma_2 \circ \delta_{L_3 M_2}) \left(\frac{1}{6} Z_{32} y Z_{31} z Z_{21} x \partial_{21} \partial_{31}(v) - \frac{1}{4} Z_{32} y Z_{31} z Z_{21} x \partial_{21}^{(2)} \partial_{32}(v) \right) \\ = \sigma_2 \left(-\frac{1}{6} Z_{21} x Z_{21} x \partial_{32}^{(2)} \partial_{21} \partial_{31}(v) \right) + \frac{1}{6} Z_{32} y Z_{21}^{(2)} x \partial_{32} \partial_{21} \partial_{31}(v) + \\ \sigma_2 \left(\frac{1}{6} Z_{32} y Z_{32} y \partial_{21}^{(2)} \partial_{21} \partial_{31}(v) \right) + \frac{1}{6} Z_{32} y Z_{31} z \partial_{21} \partial_{21} \partial_{31}(v) + \\ \sigma_2 \left(\frac{1}{4} Z_{21} x Z_{21} x \partial_{32}^{(2)} \partial_{21}^{(2)} \partial_{32}(v) \right) - \frac{1}{4} Z_{32} y Z_{21}^{(2)} x \partial_{32} \partial_{21}^{(2)} \partial_{32}(v) + \\ \sigma_2 \left(\frac{1}{4} Z_{32} y Z_{32} y \partial_{21}^{(2)} \partial_{21}^{(2)} \partial_{32}(v) \right) - \frac{1}{4} Z_{32} y Z_{31} z \partial_{21} \partial_{21}^{(2)} \partial_{32}(v) \\ = \frac{1}{3} Z_{32} y Z_{21}^{(2)} x \partial_{31}^{(2)}(v) - \frac{1}{2} Z_{32} y Z_{21}^{(2)} x \partial_{21}^{(2)} \partial_{32}^{(2)}(v) - \frac{3}{4} Z_{32} y Z_{31} z \partial_{21}^{(3)} \partial_{32}(v) - \frac{1}{12} Z_{32} y Z_{21}^{(2)} x \partial_{21} \partial_{32} \partial_{31}(v) + \\ \frac{1}{3} Z_{32} y Z_{31} z \partial_{21}^{(2)} \partial_{31}(v).$$

- $(\delta_{M_3 L_2} + \sigma_2 \circ \delta_{M_3 M_2}) (Z_{32}^{(2)} y Z_{32} y Z_{21}^{(11)} x(v));$ where $v \in D_{20} \otimes D_0 \otimes D_0$
 $= \sigma_2 (3 Z_{32}^{(3)} y Z_{21}^{(11)} x(v) - Z_{32}^{(2)} y Z_{21}^{(11)} x \partial_{32}(v) - \sigma_2 (Z_{32}^{(2)} y Z_{21}^{(10)} x \partial_{31}(v) + Z_{32}^{(2)} y Z_{32} y \partial_{21}^{(11)}(v))$
 $= \frac{1}{45} Z_{32} y Z_{21}^{(2)} x \partial_{21}^{(7)} \partial_{31}^{(2)}(v) + \frac{1}{10} Z_{32} y Z_{31} z \partial_{21}^{(9)} \partial_{31}(v).$

And

$$(\delta_{L_3 L_2} + \sigma_2 \circ \delta_{L_3 M_2}) \left(\frac{1}{90} Z_{32} y Z_{31} z Z_{21} x \partial_{21}^{(8)} \partial_{31}(v) \right) \\ = \sigma_2 \left(-\frac{1}{90} Z_{21} x Z_{21} x \partial_{32}^{(2)} \partial_{21}^{(8)} \partial_{31}(v) \right) + \frac{1}{90} Z_{32} y Z_{21}^{(2)} x \partial_{32} \partial_{21}^{(8)} \partial_{31}(v) - \\ \sigma_2 \left(\frac{1}{90} Z_{32} y Z_{32} y \partial_{21}^{(2)} \partial_{21}^{(8)} \partial_{31}(v) \right) + \frac{1}{90} Z_{32} y Z_{31} z \partial_{21} \partial_{21}^{(8)} \partial_{31}(v) \\ = \frac{1}{45} Z_{32} y Z_{21}^{(2)} x \partial_{21}^{(7)} \partial_{31}^{(2)}(v) + \frac{1}{10} Z_{32} y Z_{31} z \partial_{21}^{(9)} \partial_{31}(v).$$

- $(\delta_{M_3 L_2} + \sigma_2 \circ \delta_{M_3 M_2}) (Z_{32} y Z_{32}^{(2)} y Z_{21}^{(4)} x(v));$ where $v \in D_{13} \otimes D_7 \otimes D_0$
 $= \sigma_2 (3 Z_{32}^{(3)} y Z_{21}^{(4)} x(v) - Z_{32} y Z_{21}^{(4)} x \partial_{32}^{(2)}(v) - Z_{32} y Z_{21}^{(3)} x \partial_{32} \partial_{31}(v) - Z_{32} y Z_{21}^{(2)} x \partial_{31}^{(2)}(v) + Z_{32} y Z_{32}^{(2)} y \partial_{21}^{(4)}(v))$
 $= -\frac{2}{3} Z_{32} y Z_{21}^{(2)} x \partial_{21}^{(2)} \partial_{32}^{(2)}(v) - Z_{32} y Z_{31} z \partial_{21}^{(3)} \partial_{32}(v) - \frac{1}{3} Z_{32} y Z_{21}^{(2)} x \partial_{21} \partial_{32} \partial_{31}(v).$

And

$$\begin{aligned}
 & (\delta_{L_3L_2} + \sigma_2 \circ \delta_{L_3M_2}) \left(-\frac{1}{3} Z_{32} \psi Z_{31} z Z_{21} x \partial_{21}^{(2)} \partial_{32}(v) \right) \\
 &= \sigma_2 \left(\frac{1}{3} Z_{21} x Z_{21} x \partial_{32}^{(2)} \partial_{21}^{(2)} \partial_{32}(v) \right) - \frac{1}{3} Z_{32} \psi Z_{21} x \partial_{32} \partial_{21}^{(2)} \partial_{32}(v) + \\
 &\quad \sigma_2 \left(\frac{1}{3} Z_{32} \psi Z_{32} \psi \partial_{21}^{(2)} \partial_{21}^{(2)} \partial_{32}(v) \right) - \frac{1}{3} Z_{32} \psi Z_{31} z \partial_{21} \partial_{21}^{(2)} \partial_{32}(v) \\
 &= -\frac{2}{3} Z_{32} \psi Z_{21} x \partial_{21}^{(2)} \partial_{32}^{(2)}(v) - Z_{32} \psi Z_{31} z \partial_{21}^{(3)} \partial_{32}(v) - \frac{1}{3} Z_{32} \psi Z_{21} x \partial_{21} \partial_{32} \partial_{31}(v).
 \end{aligned}$$

- $(\delta_{M_3L_2} + \sigma_2 \circ \delta_{M_3M_2}) (Z_{32} \psi Z_{32}^{(2)} \psi Z_{21}^{(11)} x(v));$ where $v \in D_{20} \otimes D_0 \otimes D_0$
 $= \sigma_2 (3 Z_{32}^{(3)} \psi Z_{21}^{(11)} x(v)) - Z_{32} \psi Z_{21}^{(11)} x \partial_{32}^{(2)}(v) - Z_{32} \psi Z_{21}^{(10)} x \partial_{32} \partial_{31}(v) - \sigma_2 (-Z_{32} \psi Z_{21}^{(9)} x \partial_{31}^{(2)}(v) +$
 $Z_{32} \psi Z_{32}^{(2)} \psi \partial_{21}^{(11)}(v)) = \frac{3}{108} Z_{32} \psi Z_{21}^{(2)} x \partial_{21}^{(7)} \partial_{31}^{(2)}(v) - \frac{1}{36} Z_{32} \psi Z_{21}^{(2)} x \partial_{21}^{(7)} \partial_{31}^{(2)}(v) = 0.$

- $(\delta_{M_3L_2} + \sigma_2 \circ \delta_{M_3M_2}) (Z_{32}^{(3)} \psi Z_{21}^{(4)} x Z_{21} x(v));$ where $v \in D_{14} \otimes D_6 \otimes D_0$
 $= \sigma_2 (Z_{21}^{(4)} x Z_{21} x \partial_{32}^{(2)}(v) + Z_{21}^{(3)} x Z_{21} x \partial_{32}^{(2)} \partial_{31}(v) + Z_{21}^{(2)} x Z_{21} x \partial_{32} \partial_{31}^{(2)}(v) +$
 $Z_{21} x Z_{21} x \partial_{31}^{(3)}(v) - 5 Z_{32}^{(3)} \psi Z_{21}^{(5)} x(v) + Z_{32}^{(3)} \psi Z_{21}^{(4)} x \partial_{21}(v))$
 $= -\frac{2}{9} Z_{32} \psi Z_{21}^{(2)} x \partial_{21} \partial_{31}^{(2)}(v) - \frac{1}{9} Z_{32} \psi Z_{21}^{(2)} x \partial_{21}^{(3)} \partial_{32}^{(2)}(v) -$
 $\frac{2}{9} Z_{32} \psi Z_{31} z \partial_{21}^{(4)} \partial_{32}(v) - \frac{1}{6} Z_{32} \psi Z_{21}^{(2)} x \partial_{21}^{(2)} \partial_{32} \partial_{31}(v) - \frac{1}{3} Z_{32} \psi Z_{31} z \partial_{21}^{(3)} \partial_{31}(v).$

And

$$\begin{aligned}
 & (\delta_{L_3L_2} + \sigma_2 \circ \delta_{L_3M_2}) \left(\frac{-1}{9} Z_{32} \psi Z_{31} z Z_{21} x \partial_{21}^{(2)} \partial_{31}(v) - \frac{1}{18} Z_{32} \psi Z_{31} z Z_{21} x \partial_{21}^{(3)} \partial_{32}(v) \right) \\
 &= \sigma_2 \left(\frac{1}{9} Z_{21} x Z_{21} x \partial_{32}^{(2)} \partial_{21}^{(2)} \partial_{31}(v) \right) - \frac{1}{9} Z_{32} \psi Z_{21}^{(2)} x \partial_{32} \partial_{21}^{(2)} \partial_{31}(v) + \\
 &\quad \sigma_2 \left(\frac{1}{9} Z_{32} \psi Z_{32} \psi \partial_{21}^{(2)} \partial_{21}^{(2)} \partial_{31}(v) \right) - \frac{1}{9} Z_{32} \psi Z_{31} z \partial_{21} \partial_{21}^{(2)} \partial_{31}(v) + \\
 &\quad \sigma_2 \left(\frac{1}{18} Z_{21} x Z_{21} x \partial_{32}^{(2)} \partial_{21}^{(3)} \partial_{32}(v) \right) - \frac{1}{18} Z_{32} \psi Z_{21}^{(2)} x \partial_{32} \partial_{21}^{(3)} \partial_{32}(v) + \\
 &\quad \sigma_2 \left(\frac{1}{18} Z_{32} \psi Z_{32} \psi \partial_{21}^{(2)} \partial_{21}^{(3)} \partial_{32}(v) \right) - \frac{1}{18} Z_{32} \psi Z_{31} z \partial_{21} \partial_{21}^{(3)} \partial_{32}(v) \\
 &= -\frac{2}{9} Z_{32} \psi Z_{21}^{(2)} x \partial_{21} \partial_{31}^{(2)}(v) - \frac{1}{9} Z_{32} \psi Z_{21}^{(2)} x \partial_{21}^{(3)} \partial_{32}^{(2)}(v) - \\
 &\quad \frac{2}{9} Z_{32} \psi Z_{31} z \partial_{21}^{(4)} \partial_{32}(v) - \frac{1}{6} Z_{32} \psi Z_{21}^{(2)} x \partial_{21}^{(2)} \partial_{32} \partial_{31}(v) - \frac{1}{3} Z_{32} \psi Z_{31} z \partial_{21}^{(3)} \partial_{31}(v).
 \end{aligned}$$

- $(\delta_{M_3L_2} + \sigma_2 \circ \delta_{M_3M_2}) (Z_{32}^{(3)} \psi Z_{21}^{(10)} x Z_{21}^{(1)} x(v));$ where $v \in D_{20} \otimes D_0 \otimes D_0$
 $= Z_{21}^{(10)} x Z_{21} x \partial_{32}^{(3)}(v) + Z_{21}^{(9)} x Z_{21} x \partial_{32}^{(2)} \partial_{31}(v) + Z_{21}^{(8)} x Z_{21} x \partial_{32} \partial_{31}^{(2)}(v) +$
 $\sigma_2 (Z_{21}^{(7)} x Z_{21} x \partial_{31}^{(3)}(v) - 11 Z_{32}^{(3)} \psi Z_{21}^{(11)} x(v) + Z_{32}^{(3)} \psi Z_{21}^{(10)} x \partial_{21}(v)) = 0.$

- $(\delta_{M_3L_2} + \sigma_2 \circ \delta_{M_3M_2}) (Z_{32} \psi Z_{31} z Z_{21}^{(2)} x(v));$ where $v \in D_{12} \otimes D_7 \otimes D_1$
 $= \sigma_2 (Z_{32}^{(2)} \psi Z_{21}^{(3)} x(v) - Z_{21} x Z_{21}^{(2)} x \partial_{32}^{(2)}(v) + Z_{32} \psi Z_{21}^{(3)} x \partial_{32}(v) -$
 $Z_{32} \psi Z_{32} \psi \partial_{21}^{(3)}(v)) + Z_{32} \psi Z_{31} z \partial_{21}^{(2)}(v) = \frac{1}{3} Z_{32} \psi Z_{21}^{(2)} x \partial_{31}(v) + \frac{2}{3} Z_{32} \psi Z_{31} z \partial_{21}^{(2)}(v) + \frac{1}{3} Z_{32} \psi Z_{21}^{(2)} x \partial_{21} \partial_{32}(v).$

And

$$\begin{aligned}
 & (\delta_{L_3L_2} + \sigma_2 \circ \delta_{L_3M_2}) \left(\frac{1}{3} Z_{32} \psi Z_{31} z Z_{21} x \partial_{21}(v) \right) \\
 &= \sigma_2 \left(-\frac{1}{3} Z_{21} x Z_{21} x \partial_{32}^{(2)} \partial_{21}(v) \right) + \frac{1}{3} Z_{32} \psi Z_{21}^{(2)} x \partial_{32} \partial_{21}(v) - \sigma_2 \left(\frac{1}{3} Z_{32} \psi Z_{32} \psi \partial_{21}^{(2)} \partial_{21}(v) \right) + \frac{1}{3} Z_{32} \psi Z_{31} z \partial_{21} \partial_{21}(v) = \\
 &\quad \frac{1}{3} Z_{32} \psi Z_{21}^{(2)} x \partial_{31}(v) + \frac{2}{3} Z_{32} \psi Z_{31} z \partial_{21}^{(2)}(v) + \frac{1}{3} Z_{32} \psi Z_{21}^{(2)} x \partial_{21} \partial_{32}(v).
 \end{aligned}$$

- $(\delta_{M_3L_2} + \sigma_2 \circ \delta_{M_3M_2}) (Z_{32} \psi Z_{31} z Z_{21}^{(3)} x(v));$ where $v \in D_{13} \otimes D_6 \otimes D_1$
 $= \sigma_2 (2 Z_{32}^{(2)} \psi Z_{21}^{(4)} x(v) - Z_{21} x Z_{21}^{(3)} x \partial_{32}^{(2)}(v) + Z_{32} \psi Z_{21}^{(4)} x \partial_{32}(v) -$

$$\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{32}\mathcal{Y}\partial_{21}^{(4)}(v) + \mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{31}\mathcal{Z}\partial_{21}^{(3)}(v) = \frac{1}{6}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{21}^{(2)}x\partial_{21}\partial_{31}(v) + \frac{1}{2}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{31}\mathcal{Z}\partial_{21}^{(3)}(v) + \frac{1}{6}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{32}(v).$$

And

$$\begin{aligned} &(\delta_{L_3L_2} + \sigma_2 \circ \delta_{L_3M_2})\left(\frac{1}{6}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{31}\mathcal{Z}\mathcal{Z}_{21}x\partial_{21}^{(2)}(v)\right) \\ &= \sigma_2\left(-\frac{1}{6}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(2)}(v)\right) + \frac{1}{6}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(2)}(v) - \\ &\quad \sigma_2\left(\frac{1}{6}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{32}\mathcal{Y}\partial_{21}^{(2)}\partial_{21}^{(2)}(v)\right) + \frac{1}{6}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{31}\mathcal{Z}\partial_{21}\partial_{21}^{(2)}(v) \\ &= \frac{1}{6}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{21}^{(2)}x\partial_{21}\partial_{31}(v) + \frac{1}{2}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{31}\mathcal{Z}\partial_{21}^{(3)}(v) + \frac{1}{6}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{32}(v). \end{aligned}$$

- $(\delta_{M_3L_2} + \sigma_2 \circ \delta_{M_3M_2})\left(\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{31}\mathcal{Z}\mathcal{Z}_{21}^{(4)}x(v)\right)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_5 \otimes \mathcal{D}_1$
 $= \sigma_2\left(3\mathcal{Z}_{32}^{(2)}\mathcal{Y}\mathcal{Z}_{21}^{(5)}x(v) - \mathcal{Z}_{21}x\mathcal{Z}_{21}^{(4)}x\partial_{32}^{(2)}(v) + \mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{21}^{(5)}x\partial_{32}(v) - \right.$
 $\left. \mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{32}\mathcal{Y}\partial_{21}^{(5)}(v) + \mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{31}\mathcal{Z}\partial_{21}^{(4)}(v) = \frac{1}{10}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{31}(v) + \frac{2}{5}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{31}\mathcal{Z}\partial_{21}^{(4)}(v) + \frac{1}{10}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{32}(v)\right).$

And

$$\begin{aligned} &(\delta_{L_3L_2} + \sigma_2 \circ \delta_{L_3M_2})\left(\frac{1}{10}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{31}\mathcal{Z}\mathcal{Z}_{21}x\partial_{21}^{(3)}(v)\right) \\ &= \sigma_2\left(-\frac{1}{10}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(3)}(v)\right) + \frac{1}{10}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(3)}(v) - \sigma_2\left(\frac{1}{10}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{32}\mathcal{Y}\partial_{21}^{(2)}\partial_{21}^{(3)}(v)\right) + \frac{1}{10}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{31}\mathcal{Z}\partial_{21}\partial_{21}^{(3)}(v) \\ &= \frac{1}{10}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{31}(v) + \frac{2}{5}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{31}\mathcal{Z}\partial_{21}^{(4)}(v) + \frac{1}{10}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{32}(v). \end{aligned}$$

- $(\delta_{M_3L_2} + \sigma_2 \circ \delta_{M_3M_2})\left(\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{31}\mathcal{Z}\mathcal{Z}_{21}^{(9)}x(v)\right)$; where $v \in \mathcal{D}_{19} \otimes \mathcal{D}_0 \otimes \mathcal{D}_1$
 $= \sigma_2\left(8\mathcal{Z}_{32}^{(2)}\mathcal{Y}\mathcal{Z}_{21}^{(10)}x(v) - \mathcal{Z}_{21}x\mathcal{Z}_{21}^{(9)}x\partial_{32}^{(2)}(v) + \mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{21}^{(10)}x\partial_{32}(v) - \right.$
 $\left. \sigma_2\left(\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{32}\mathcal{Y}\partial_{21}^{(10)}(v) + \mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{31}\mathcal{Z}\partial_{21}^{(9)}(v) = \frac{1}{45}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(7)}\partial_{31}(v) + \frac{1}{5}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{31}\mathcal{Z}\partial_{21}^{(9)}(v)\right).$

And

$$\begin{aligned} &(\delta_{L_3L_2} + \sigma_2 \circ \delta_{L_3M_2})\left(\frac{1}{45}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{31}\mathcal{Z}\mathcal{Z}_{21}x\partial_{21}^{(8)}(v)\right) \\ &= \sigma_2\left(-\frac{1}{45}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(8)}(v)\right) + \frac{1}{45}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(8)}(v) - \\ &\quad \sigma_2\left(\frac{1}{45}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{32}\mathcal{Y}\partial_{21}^{(2)}\partial_{21}^{(10)}(v)\right) + \frac{1}{45}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{31}\mathcal{Z}\partial_{21}\partial_{21}^{(8)}(v) = \frac{1}{45}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(7)}\partial_{31}(v) + \frac{1}{5}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{31}\mathcal{Z}\partial_{21}^{(9)}(v). \end{aligned}$$

- $(\delta_{M_3L_2} + \sigma_2 \circ \delta_{M_3M_2})\left(\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{31}\mathcal{Z}(v)\right)$; where $v \in \mathcal{D}_{10} \otimes \mathcal{D}_{10} \otimes \mathcal{D}_0$
 $= \sigma_2\left(2\mathcal{Z}_{32}^{(2)}\mathcal{Y}\mathcal{Z}_{31}\mathcal{Z}(v) - 2\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{31}\mathcal{Z}(v) + \mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{32}\mathcal{Y}\partial_{31}(v)\right) = 0.$

- $(\delta_{M_3L_2} + \sigma_2 \circ \delta_{M_3M_2})\left(\mathcal{Z}_{32}^{(2)}\mathcal{Y}\mathcal{Z}_{31}\mathcal{Z}\mathcal{Z}_{21}^{(2)}x(v)\right)$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_8 \otimes \mathcal{D}_0$
 $= \sigma_2\left(-\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(2)}x\partial_{32}^{(2)}(v) + \mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(3)}x\partial_{32}(v) + \mathcal{Z}_{32}^{(2)}\mathcal{Y}\mathcal{Z}_{32}\mathcal{Y}\partial_{21}^{(3)}(v) + \right.$
 $\left. \mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{31}\mathcal{Z}\partial_{21}^{(2)}(v)\right) = \frac{1}{3}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{31}(v) + \frac{1}{3}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{31}\mathcal{Z}\partial_{21}\partial_{31}(v).$

And

$$\begin{aligned} &(\delta_{L_3L_2} + \sigma_2 \circ \delta_{L_3M_2})\left(\frac{1}{3}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{31}\mathcal{Z}\mathcal{Z}_{21}x\partial_{31}(v)\right) \\ &= \sigma_2\left(-\frac{1}{3}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{31}(v)\right) + \frac{1}{3}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{31}(v) - \\ &\quad \sigma_2\left(\frac{1}{3}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{32}\mathcal{Y}\partial_{21}^{(2)}\partial_{31}(v)\right) + \frac{1}{3}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{31}\mathcal{Z}\partial_{21}\partial_{31}(v) = \frac{1}{3}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{31}(v) + \frac{1}{3}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{31}\mathcal{Z}\partial_{21}\partial_{31}(v). \end{aligned}$$

- $(\delta_{M_3L_2} + \sigma_2 \circ \delta_{M_3M_2})\left(\mathcal{Z}_{32}^{(2)}\mathcal{Y}\mathcal{Z}_{31}\mathcal{Z}\mathcal{Z}_{21}^{(10)}x(v)\right)$; where $v \in \mathcal{D}_{20} \otimes \mathcal{D}_0 \otimes \mathcal{D}_0$
 $= \sigma_2\left(8\mathcal{Z}_{32}^{(3)}\mathcal{Y}\mathcal{Z}_{21}^{(11)}x(v) - \mathcal{Z}_{21}x\mathcal{Z}_{21}^{(10)}x\partial_{32}^{(3)}(v) + \mathcal{Z}_{32}^{(2)}\mathcal{Y}\mathcal{Z}_{21}^{(11)}x\partial_{32} - \right.$
 $\left. \sigma_2\left(\mathcal{Z}_{32}^{(2)}\mathcal{Y}\mathcal{Z}_{32}\mathcal{Y}\partial_{21}^{(11)}(v) - \mathcal{Z}_{32}^{(2)}\mathcal{Y}\mathcal{Z}_{31}\mathcal{Z}\partial_{21}^{(10)}(v)\right) = \frac{2}{27}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(7)}\partial_{31}^{(2)}(v) + \frac{1}{3}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{31}\mathcal{Z}\partial_{21}^{(9)}\partial_{31}(v).\right.$

And

$$\begin{aligned}
 & (\delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2}) \left(\frac{1}{27} \mathcal{Z}_{32} \mathcal{Y} \mathcal{Z}_{31} \mathcal{Z} \mathcal{Z}_{21} x \partial_{21}^{(8)} \partial_{31}(v) \right) \\
 &= \sigma_2 \left(-\frac{1}{27} \mathcal{Z}_{21} x \mathcal{Z}_{21} x \partial_{32}^{(2)} \partial_{21}^{(8)} \partial_{31}(v) \right) + \frac{1}{27} \mathcal{Z}_{32} \mathcal{Y} \mathcal{Z}_{21}^{(2)} x \partial_{32} \partial_{21}^{(7)} \partial_{31}(v) - \\
 & \quad \sigma_2 \left(\frac{1}{27} \mathcal{Z}_{32} \mathcal{Y} \mathcal{Z}_{32} \mathcal{Y} \partial_{21}^{(2)} \partial_{21}^{(8)} \partial_{31}(v) \right) + \frac{9}{27} \mathcal{Z}_{32} \mathcal{Y} \mathcal{Z}_{31} \mathcal{Z} \partial_{21} \partial_{21}^{(9)} \partial_{31}(v) \\
 &= \frac{2}{27} \mathcal{Z}_{32} \mathcal{Y} \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(7)} \partial_{31}^{(2)}(v) + \frac{1}{3} \mathcal{Z}_{32} \mathcal{Y} \mathcal{Z}_{31} \mathcal{Z} \partial_{21}^{(9)} \partial_{31}(v).
 \end{aligned}$$

Eventually, we define the boundary maps in the complex:

$$0 \longrightarrow \mathcal{L}_3 \xrightarrow{\partial_3} \mathcal{L}_2 \xrightarrow{\partial_2} \mathcal{L}_1 \xrightarrow{\partial_1} \mathcal{L}_0; \tag{4}$$

where ∂_1 is the operation of indicated polarization operators, ∂_1 , ∂_2 and ∂_3 defined as follows:

- $\partial_1(\mathcal{Z}_{21}x(v)) = \partial_{21}(v)$; where $v \in \mathcal{D}_{10} \otimes \mathcal{D}_7 \otimes \mathcal{D}_3$.
- $\partial_1(\mathcal{Z}_{32}\mathcal{Y}(v)) = \partial_{32}(v)$; where $v \in \mathcal{D}_9 \otimes \mathcal{D}_9 \otimes \mathcal{D}_2$.
- $\partial_2(\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{21}^{(2)}x(v)) = \frac{1}{2} \mathcal{Z}_{21}x\partial_{21}\partial_{32}(v) + \mathcal{Z}_{21}x\partial_{31}(v) - \mathcal{Z}_{32}\mathcal{Y}\partial_{21}^{(2)}(v)$; where $v \in \mathcal{D}_{11} \otimes \mathcal{D}_7 \otimes \mathcal{D}_2$.
- $\partial_2(\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{31}\mathcal{Z}(v)) = \frac{1}{2} \mathcal{Z}_{32}\mathcal{Y}\partial_{32}\partial_{21}(v) - \mathcal{Z}_{21}x\partial_{32}^{(2)}(v) - \mathcal{Z}_{32}\mathcal{Y}\partial_{31}(v)$; where $v \in \mathcal{D}_{10} \otimes \mathcal{D}_9 \otimes \mathcal{D}_1$.
- $\partial_3(\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{31}\mathcal{Z}\mathcal{Z}_{21}x(v)) = \mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{21}^{(2)}x\partial_{32}(v) + \mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{31}\mathcal{Z}\partial_{21}(v)$; where $v \in \mathcal{D}_{11} \otimes \mathcal{D}_8 \otimes \mathcal{D}_1$.

Theorem (3.5):

The complex (3.4) is exact and in characteristic-zero gives a resolution of $K_{(9,8,3)}(\mathcal{F})$.

Proof:

First, we prove the exactness of the complex

$$0 \longrightarrow \mathcal{L}_3 \xrightarrow{\partial_3} \mathcal{L}_2 \xrightarrow{\partial_2} \mathcal{L}_1$$

Since one component of the map ∂_3 is a diagonalization of \mathcal{D}_2 into $\mathcal{D}_1 \otimes \mathcal{D}_1$ it is clear that ∂_3 is injective. To prove the exactness at \mathcal{L}_2 .

For this, we need to show that:

If $v \in \ker(\partial_2)$ then $\exists w \in \mathcal{L}_3$ such that $\partial_3(w) = v$.

If $\partial_2(v) = 0$ then $\exists (a, b) \in \mathcal{L}_3 \oplus \mathcal{M}_3$ such that

$\delta(a, b) = (v, 0) \in \mathcal{L}_2 \oplus \mathcal{M}_2$, but

$\delta(a, b) = \delta_{\mathcal{L}_3\mathcal{L}_2}(a) + \delta_{\mathcal{L}_3\mathcal{M}_2}(a) + \delta_{\mathcal{M}_2\mathcal{L}_2}(b) + \delta_{\mathcal{M}_3\mathcal{M}_2}(b)$. So we get:

$$\delta_{\mathcal{L}_3\mathcal{L}_2}(a) + \delta_{\mathcal{M}_3\mathcal{L}_2}(b) = v \tag{5}$$

and

$$\delta_{\mathcal{L}_3\mathcal{M}_2}(a) + \delta_{\mathcal{M}_3\mathcal{M}_2}(b) = 0 \tag{6}$$

Now if $w = a + \sigma_3(b)$ we can see that $\partial_3(w) = v$ in fact

$\partial_3(a) = \delta_{\mathcal{L}_3\mathcal{L}_2}(a) + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2}(a)$, and

$\partial_3(\sigma_3(b)) = \delta_{\mathcal{M}_3\mathcal{L}_2}(b) + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}(b)$, so

$$\begin{aligned}
 \partial_3(a + \sigma_3(b)) &= \partial_3(a) + \partial_3(\sigma_3(b)) \\
 &= \delta_{\mathcal{L}_3\mathcal{L}_2}(a) + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2}(a) + \delta_{\mathcal{M}_2\mathcal{L}_2}(b) + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}(b) \\
 &= \delta_{\mathcal{L}_3\mathcal{L}_2}(a) + \delta_{\mathcal{M}_3\mathcal{L}_2}(b) + \sigma_2 \circ \left(\delta_{\mathcal{L}_3\mathcal{M}_2}(a) + \delta_{\mathcal{M}_3\mathcal{M}_2}(b) \right).
 \end{aligned}$$

Hence from (1) and (2), we get $\partial_3(w) = v$; where $w = a + \sigma_3(b)$.

This proves the exactness at \mathcal{L}_2 .

As the same way we can prove the exactness at \mathcal{L}_1 .

Finally, we get the complex:

$$0 \longrightarrow \mathcal{L}_3 \xrightarrow{\partial_3} \mathcal{L}_2 \xrightarrow{\partial_2} \mathcal{L}_1 \xrightarrow{\partial_1} \mathcal{L}_0 \longrightarrow \mathcal{K}_{(9,8,3)}(\mathcal{F}) \longrightarrow 0,$$

is exact.

Conflicts Of Interest

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