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Research Article

Existence and Uniqueness Theorem of Multi-Dimensional Integro-Differential Equations With Fractional Differointegrations

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ABSTRACT

The aim of this paper is to establish the existence and uniqueness theorems of multi-dimensional conformable fractional partial integro-differential equations theorem for generalized contraction mappings with respect to w-distances in complete metric spaces.

1. INTRODUCTION

Fractional calculus was introduced by mathematicians like Leibniz and Euler in the 18th century, but it was not until the 20th century that it began to be studied more systematically. In the 1970s, researchers like Samko, Kilbas, and Marichev developed a theory of fractional integrals and derivatives that was based on the Riemann-Liouville definition of fractional derivatives. Fractional calculus was introduced by mathematicians like Leibniz and Euler in the 18th century, but it was not until the 20th century that it began to be studied more systematically. In the 1970s, researchers like Samko, Kilbas, and Marichev developed a theory of fractional integrals and derivatives that was based on the Riemann-Liouville definition of fractional derivatives [4,5].

In recent years, there has been growing interest in extending fractional calculus to higher dimensions, and this has led to the development of the theory of CFPIDEs. These equations are a generalization of fractional partial differential equations, and they involve both fractional derivatives and fractional integrals. The first Existence and Uniqueness Theorem for CFPIDEs was established in 2019 by Jafari et al. In their work, they proved that under certain conditions, CFPIDEs have a unique solution that can be expressed in terms of a series solution. The conditions they imposed were based on a new concept of conformable fractional derivative, which is a modification of the Riemann-Liouville fractional derivative that is better suited for higher-dimensional problems. Since the publication of Jafari et al.'s result, there has been a growing interest in the study of CFPIDEs and their applications in fields like physics, engineering, and finance. Researchers have continued to refine the theory and develop new techniques for solving CFPIDEs, and it is likely that this area of mathematics will continue to be an active area of research for years to come [6,7,8].

Various phenomena's of viscoelastic its, diffusion procedures, relaxation vibrations, electrochemistry, etc. are successfully described by fractional differential equations (FDEs) and therefor The researchers tried to suggest several types of fractional operators to describe more accurately these phenomena's since fractional order deferential equations are generalization of integer order deferential equations to non-integer order ones. The fractional calculus was bounded up with fractional integrals

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obtained by iterating an integral to get the $n-th$ order integral and after that replacing n by any number, and then by using the classical method the corresponding derivatives were defined, proposed a new derivative with real orders, and hence different definitions of fractional integrals and derivatives are proposed and In this work the conformable fractional calculus will be considered. Which is due to its well-behaved properties and the close relationship with first order derivative, conformable derivatives and integrals and has exerted a tremendous fascination on researchers [1, 2].

The most well-known fixed-point results in the metrical fixed point theory are on based Banach's contraction mapping principle, Moreover, this principle has many applications not only of the various branches in mathematical topics, but also in economics, chemistry, biology, computer science, engineering, and others. Based on the mentioned impact, it was developed extensively by several researchers [3]. This enables a researcher to choose the most suitable operator in order to describe the dynamics in a real world problem.

2. THEORETICAL BACKGROUND FOR BANACH FIXED POINT THEOREM

This section introduces some basic concepts that are necessary for establishing the theorem on the existence and uniqueness for the solution and to find sufficient conditions that satisfies the Lipschitz condition. The main aim of this section is to provide some necessary definitions and a theorem including Banach fixed point theorem, which are used throughout this Paper. Now, we will start with the basic concepts related to this work, in which more elementary concepts of undergraduate study will be omitted.

Definition 2.1. [32] Let $(X, \|\cdot\|)$ be a normed space and $T: X \to X$ be a mapping. A point $x \in X$ for which $Tx = x$ is called a fixed point of T.

Definition 2.2. [32] The mapping T on a normed space $(X, \|\cdot\|)$ is called contractive if there is a non-negative real number $c \in (0,1)$, such that:

$$\|Tx_1 - Tx_2\| \leq \|x_1 - x_2\|, \text{ for all } x_1, x_2 \in X$$

where c is called the contractivity factor.

The theorem that is used in upcoming chapters, is the so called Banach fixed point theorem which is stated next.

Theorem 2.3 [9] Let $(X, \|\cdot\|)$ be a complete normed space and let the mapping $T : X \to X$ be a contraction mapping, then T has exactly one fixed point.

An additional definition which is also necessary in the proof the existence and uniqueness of solution of the considered IDE in this work is the next definition of Lipschitzian function.

Definition 2.4. [13] Let $f: A \to \mathbb{R}^n$ be a continuously differentiable function over an interval $[a, b]$, where $A \subset \mathbb{R}^n$ is said to satisfy Liptischitz condition if there exists a constant $L > 0$ (dependent on both function and the interval), such that:

$$\|f(a) - f(b)\| \leq L\|a - b\|$$

for every pair of points $a, b \in A$.

Remark 2.5. The space $C_t^{m-1}([a, b] \times [0, T])$ will be used to denote the Banach space of all continuous real valued functions u defined on $[a, b] \times [0, T]$ with continuous m^{th} order partial derivatives with respect to t.

3. EXISTENCE AND UNIQUENESS OF MULTI-DIMENSIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH FRACTIONAL DERIVATIVE

One of the most important tasks in this Paper is to find the approximate solution of multi-dimensional integro-differential equations to satisfy the existence and uniqueness theorem. As we said above, we shall use the fixed-point principle based on Banach fixed point theorem or the contraction mapping. In this section, we will state and prove the existence theorem of the multi-dimensional integro-differential equations with fractional derivative that has the form:

$$T_x^{\alpha} u(x, y) = g(x, y) + \int_a^x \int_b^y k(x, y, s, t, u(x, y))dsdt, x \in [a, b], t \in [0, T] \tag{1}$$

where k is the kernel function, g is a given function and u is the unknown real valued function that need to be determined, for $a \leq x \leq b, 0 \leq t \leq T, a, b, t$ and $T \in$ R$, m - 1 < \alpha, \beta, \gamma \leq m, m \in \mathbb{N}$ and D_t^{α} denote the Caputo fractional order derivative.

Theorem 3.1. Consider the multi-dimensional integro-differential equation given by equation (1) over $x \in [a, b], y \in [c, d]$ and suppose that the kernel K satisfies Lipschitz condition with respect to u and constant L, such that $L < \frac{\alpha+1}{(b-a)^{\alpha+1}(d-s)}$. Then equation (1) has a unique solution.

Proof. Apply $T_x^{\alpha}, \alpha > 0$ to the both sides of equation (1), yield to:

$$T_x^{\alpha}u(x,y) = g(x,y) + \int_a^x \int_c^y K(x,y,s,t,u(s,t))dtds \tag{2}$$

$$I_x^{\alpha}T_x^{\alpha}u(x,y) = I_x^{\alpha}g(x,y) + I_x^{\alpha}\int_a^x \int_c^y K(x,y,s,t,u(s,t))dtds$$

implies that:

$$u(x,y) = u_0(x,y) + I_x^{\alpha}g(x,y) + \int_a^x \int_a^z \int_c^y (z-x)^{\alpha-1}K(x,y,s,t,u(s,t))dtdsdz \tag{3}$$
$$= Au(x,y)$$

where A is an integral operator.

Take u_1, u_2 be any two functions, then to prove that A is a contraction, we have:

$$\|Au_1 - Au_2\| = \|u_0(x,y) + I_x^{\alpha}g(x,y) + \int_a^x \int_a^z \int_c^y (z-x)^{\alpha-1}K(x,y,s,t,u_1(s,t))dtdsdz -$$
$$u_0(x,y) - I_x^{\alpha}g(x,y) - \int_a^x \int_a^y \int_c^y (z-x)^{\alpha-1}K(x,y,s,t,u_2(s,t))dtdsdz\|$$
$$\le \int_a^x \int_a^z \int_c^y (z-x)^{\alpha-1}\|K(x,y,s,t,u_1(s,t)) - K(x,y,s,t,u_2(s,t))\|dtdsdz$$
$$\le L\int_a^x \int_a^z \int_c^y (z-x)^{\alpha-1}\|u_1(s,t) - u_2(s,t)\|dtdsdz$$
$$\le L\|u_1 - u_2\|\int_a^x \int_a^z \int_c^y (z-x)^{\alpha-1}dtdsdz$$
$$\le L\|u_1 - u_2\|\frac{(x-a)^{\alpha-1}(y-c)}{\alpha+1}$$

Hence:

$$\|Au_1 - Au_2\| \le L\|u_1 - u_2\|\frac{(x-a)^{\alpha-1}(y-c)}{\alpha+1}, x \in [a,b], y \in [c,d] \tag{4}$$

Taking the supremum value over x and y, imples:

$$\|Au_1 - Au_2\| \le \frac{L(b-a)^{\alpha-1}(d-c)}{\alpha+1}\|u_1 - u_2\|$$

Since $\frac{L(b-a)^{\alpha-1}(d-c)}{\alpha+1} < 1$, because $L < \frac{\alpha+1}{(b-a)^{\alpha-1}(d-c)}$ Therefore, A is a contractive mapping and hence by Banach fixed point theorem A has a unique fixed point which means that the integro-differential equation (1) has a unique solution.

4. EXISTENCE AND UNIQUENESS OF MULTI-DIMENSIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH FRACTIONAL INTEGRALS

In this section, the statement and the proof of the existence and uniqueness theorem of multi-dimensional integro-differential equations with fractional integrals that given by:

$$\frac{\partial u(x,y)}{\partial x} = g(x,y) + I_x^{\beta}I_y^{\gamma}k\big(x,y,s,t,u(x,y)\big)dsdt, x \in [a,b], t \in [0,T] \tag{5}$$

where K is the kernel function, g is a given function and u is an unknown real valued function that need to be determined, for $a \le x \le b, 0 \le t \le T, a, b, t$ and $T \in$ R$, m-1 < \alpha, \beta, \gamma \le m, m \in \mathbb{N}$. ${}_aI_x^{\beta}$, ${}_0I_t^{\gamma}$ denotes the Riemann-Liouville fractional order integral operators.

Theorem 4.1. Consider the multidimensional integro-differential equation of fractional order given by equation (1.24) and suppose that the kernel K satisfies Lipschitz condition with respect to u and constant L, such that $L < \frac{\beta(\beta+1)\gamma}{(b-a)^{\beta+1}(d-c)^{\gamma}}$, then equation (5) has a unique solution.

Proof. Consider equation (5), which is given by:

$$\frac{\partial u}{\partial x} = g(x,y) + I_x^{\beta}I_y^{\gamma}K(x,y,s,t,u(s,t))$$

where I_x^{β} and I_y^{γ} refers to conformal integrals of order β and γ, respectively. Take the integral with respect to x to the both sides of the above equation, which give:

$$u(x,y) = u(a,y) + \int_a^x g(z,y)dz + \int_a^x I_z^\beta I_y^\gamma K(z,y,s,t,u(s,t))dz$$

which is expanded to:

$$u(x,y) = u(a,y) + \int_a^x g(z,y)dz + \int_a^x \int_a^z (s-a)^{\beta-1} \int_c^y (t-c)^{\gamma-1} K(z,y,s,t,u(s,t))dtdsdz \quad (6)$$

Assume the integral operator:

$$Au(x,y) = u(a,y) + \int_a^x g(z,y)dz + \int_a^x \int_a^z (s-a)^{\beta-1} \int_c^y (t-c)^{\gamma-1} K(z,y,s,t,u(s,t))dtdsdz$$

then the integral equation (2) in operator form $u(x,y) = Au(x,y)$ will have a unique solution if the integral operator A is a contraction.

Let u_1 and u_2 be two integrable functions, then:

$$\|Au_1(x,y) - Au_2(x,y)\| = \|u(a,y) + \int_a^x g(z,y)dz + \int_a^x \int_a^z (s-a)^{\beta-1} \int_c^y (t-c)^{\gamma-1} K(z,y,s,t,u_1(s,t))dtdsdz - u(a,y) - \int_a^x g(z,y)dz - \int_a^x \int_a^z (s-a)^{\beta-1} \int_c^y (t-c)^{\gamma-1} K(z,y,s,t,u_2(s,t))dtdsdz\|$$

$$= \left\| \int_a^x \int_a^z \int_c^y (s-a)^{\beta-1}(t-c)^{\gamma-1}[K(z,y,s,t,u_1(s,t)) - K(z,y,s,t,u_2(s,t))]dtdsdz \right\|$$

$$\leq L\|u_1(x,y) - u_2(x,y)\| \int_a^x \int_a^z \int_c^y (s-a)^{\beta-1}(t-c)^{\gamma-1} dtdsdz \quad (7)$$

Then after carrying some calculations over inequality (3.2), we get:

$$\|Au_1 - Au_2\| \leq \frac{L(x-a)^{\beta+1}(t-c)^\gamma}{\beta(\beta+1)\gamma} \|u_1 - u_2\| \quad (8)$$

and upon taking the maximum values of x and y on the brackets, implies to:

$$\|Au_1 - Au_2\| \leq \frac{L(b-a)^{\beta+1}(d-c)^\gamma}{\beta(\beta+1)\gamma} \|u_1 - u_2\| \quad (9)$$

Since $\frac{L(b-a)^{\beta+1}(d-c)^\gamma}{\beta(\beta+1)\gamma} < 1$, because $L < \frac{\beta(\beta+1)\gamma}{(b-a)^{\beta+1}(d-c)^\gamma}$ Thu A is a contraction mapping and by Banach fixed point theorem it has a unique fixed point which is equivalent to that the integro-differential equation (5) has a unique solution.

5. Existence and Uniqueness of the Generalized Multi-Dimensional Integro-Differential Equations with Fractional Diffrointegrals

In this section, the statement and the proof of the existence and uniqueness theorem of the generalized multi-dimensional integro-differential equations with fractional diffrointegrals that given by:

$$T_t^\alpha u(x,y) = g(x,y) + I_x^\beta I_y^\gamma K(x,y,s,t,u(x,y))dsdt, \alpha \in \mathbb{R}^+, x \in [a,b], t \in [0,T] \quad (10)$$

where K is the kernel function, g is a given function and u is an unknown real valued function that need to be determined, for $a \leq x \leq b, 0 \leq t \leq T, a, b, t$ and $T \in$ R, $m-1 < \alpha, \beta, \gamma \leq m, m \in \mathbb{N}$. D_t^α denote the Caputo fractional order derivative and ${}_aI_x^\beta$, ${}_0I_t^\gamma$ denotes the Riemann-Liouville fractional order integral operators and both Caputo fractional order derivative and Riemann-Liouville fractional order integral operators.

Theorem 5.1. Consider the multidimensional integro-differential equation of fractional order given by equation (5) and suppose that the kernel K satisfies Lipschitz condition with respect to u and constant L, such that $L < \frac{\beta(\beta+1)\gamma}{(b-a)^{\beta+1}(d-c)^\gamma}$, then equation (5) has a unique solution.

Proof. Consider equation (5), which is given by:

$$\frac{\partial u}{\partial x} = g(x,y) + I_x^\beta I_y^\gamma K(x,y,s,t,u(s,t))$$

where I_x^β and I_y^γ refers to conformal integrals of order β and γ, respectively. Take the integral with respect to x to the both sides of the above equation, which give:

$$u(x,y) = u(a,y) + \int_a^x g(z,y)dz + \int_a^x I_z^\beta I_y^\gamma K(z,y,s,t,u(s,t))dz$$

which is expanded to:

$$u(x,y)=u(a,y)+\int_a^x g(z,y)dz+\int_a^x\int_a^z (s-a)^{\beta-1}\int_c^y (t-c)^{\gamma-1}K(z,y,s,t,u(s,t))dtdsdz \tag{11}$$

Assume the integral operator:
$Au(x,y)=u(a,y)+\int_a^x g(z,y)dz+\int_a^x\int_a^z (s-a)^{\beta-1}\int_c^y (t-c)^{\gamma-1}K(z,y,s,t,u(s,t))dtdsdz$
then the integral equation (2) in operator form $u(x,y)=Au(x,y)$ will have a unique solution if the integral operator A is a contraction.
Let u_1 and u_2 be two integrable functions, then:

$$\|Au_1(x,y)-Au_2(x,y)\|=\|u(a,y)+\int_a^x g(z,y)dz+\int_a^x\int_a^z (s-a)^{\beta-1}$$
$$\int_c^y (t-c)^{\gamma-1}K(z,y,s,t,u_1(s,t))dtdsdz-u(a,y)-\int_a^x g(z,y)dz-$$
$$=\|\int_a^x\int_a^z (s-a)^{\beta-1}\int_c^y (t-c)^{\gamma-1}K(z,y,s,t,u_2(s,t))dtdsdz$$
$$\le\int_c^z (s-a)^{\beta-1}(t-c)^{\gamma-1}[K(z,y,s,t,u_1(s,t))-K(z,y,s,t,u_2(s,t))]dtdsdz\|$$

Then after carrying some calculations over inequality (3), we get:

$$\|Au_1-Au_2\|\le\frac{L(x-a)^{\beta+1}(t-c)^{\gamma}}{\beta(\beta+1)\gamma}\|u_1-u_2\| \tag{12}$$

and upon taking the maximum values of x and y on the brackets, implies to:

$$\|Au_1-Au_2\|\le\frac{L(b-a)^{\beta+1}(d-c)^{\gamma}}{\beta(\beta+1)\gamma}\|u_1-u_2\| \tag{13}$$

Since $\frac{L(b-a)^{\beta+1}(d-c)^{\gamma}}{\beta(\beta+1)\gamma}<1$, because $L<\frac{\beta(\beta+1)\gamma}{(b-a)^{\beta+1}(d-c)^{\gamma}}$, then A is a contraction mapping and by Banach fixed point theorem it has a unique fixed point which is equivalent to that the integro-differential equation (5) has a unique solution.

Conflicts Of Interest

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