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Review Article

Some Expansions to The Weibull Distribution Families with Two Parameters: A Review

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ABSTRACT

This study is concerned with reviewing the different expansions of the two parameters Weibull distribution families, which are considered popular probability distributions used in many fields, such as reliability analysis, systems engineering, and fault studies. The research aims to provide a comprehensive review of the scientific literature related to these expansions, focusing on the theoretical frameworks, practical applications, and limitations facing each type of these expansions. The main points of this study are summarized in a mathematical review of methods by which the expansions were developed to improve their suitability to various data and discussing the improvements that were added to address the problems that arise when applying the traditional version of the distribution. Integrating these expansions with the exponential distribution as a baseline to discuss the performance of each expansion, a Monte Carlo simulation was presented for each expansion and choose the best expansion using some criteria this is to calculate the efficient of estimating parameters for each expansion. As for the practical application on real data, these expansions were applied to real data to decide the issues of performance accuracy in practical application, and compare the performance between different expansions using measures such as statistical accuracy and ease of interpretation.

1. INTRODUCTION

Two parameter Weibull distribution is one the most important probability distributions used applied statistics, due to its great flexibility and ability to represent many natural and industrial phenomena. This distribution characterized by its ability to describe data related to failure time, reliability analysis, and failure modeling, which makes it of great value in fields such as engineering, medicine, and the environment. However, the traditional version of this distribution may face limitations in representing data with complex or multiple patterns.

Over the past decades, many extensions to the Weibull distribution have been developed to improve its flexibility and ability to handle divers data. These extensions focus on introducing additional or combining it with other distributions to achieve a better fit. This review aims to comprehensively review these extensions, highlight the most important theoretical and practical contributions, and analyze p previous literature to understand the strengths and weaknesses of each extension.

The CDF and PDF functions express the two parameter Weibull distribution respectively as forms [1]:

$$f(x,\alpha,\beta) = \alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}} \tag{1}$$

$$f(x,\alpha,\beta) = \alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}} \tag{2}$$

These function are shown in the following figures:

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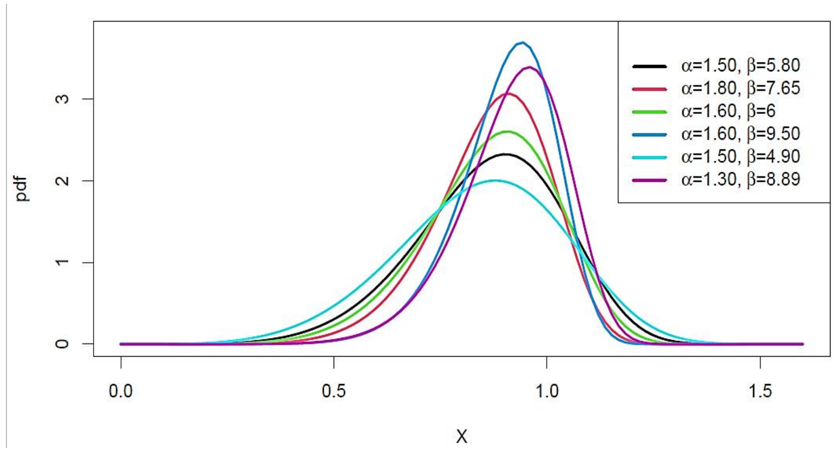
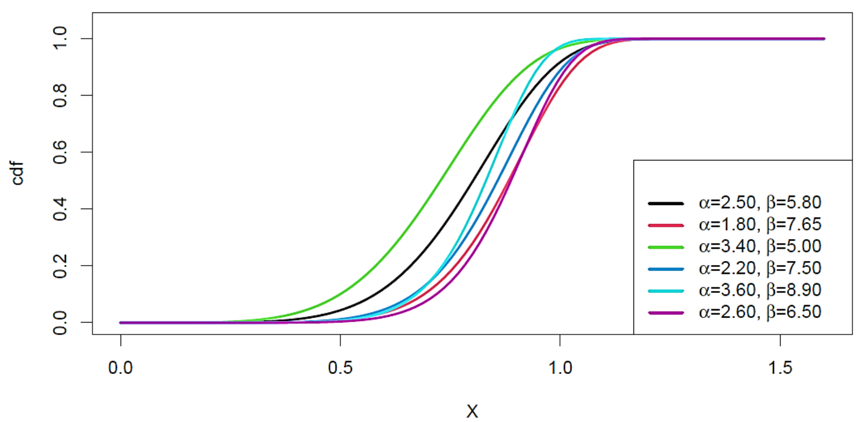


(a)



(b)

Fig. 1. plot (a) PDF and (b) CDF functions for Weibull distribution with different values for α, and β

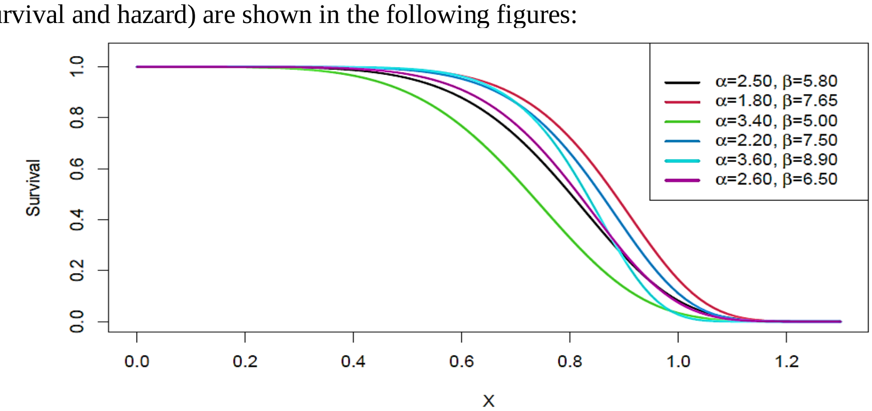
And the survival function for Weibull distribution has a form [2]:

$$S(x,\alpha,\beta) = 1 - F(x,\alpha,\beta) = e^{-\alpha x^{\beta}} \tag{3}$$

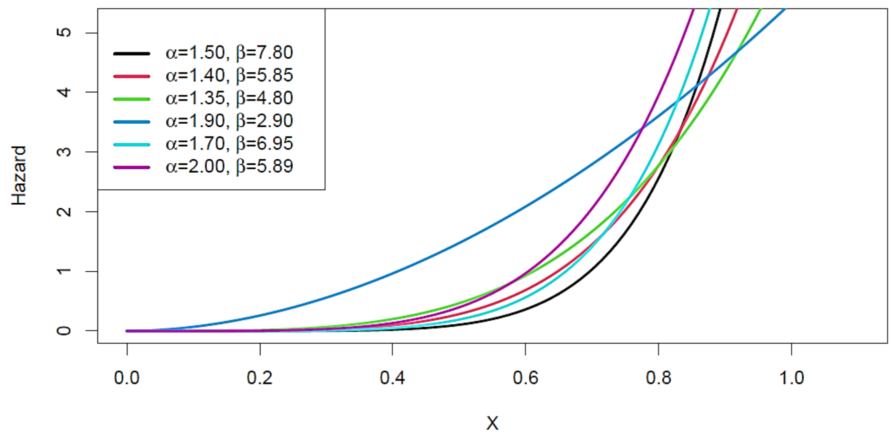
While the hazard function for Weibull distribution has a form [3]:

$$h(x,\alpha,\beta) = \frac{f(x,\alpha,\beta)}{S(x,\alpha,\beta)} = \alpha\beta x^{\beta-1} \tag{4}$$

These function (survival and hazard) are shown in the following figures:



(a)



(b)

Fig. 2. plot (a) survival and (b) hazard functions for Weibull distribution with different values for α, and β

Despite the great advances in expansions, there are clear gaps that can be highlighted:

- Many expansions add additional parameters, which increases the complexity of models and makes it difficult to estimate the parameters in practice.
- Most studies focus on theoretical aspects without providing practical applications that demonstrate the performance of the expansions on real world data.
- Complex expansions make it difficult to interpret statistical parameters and their meanings in in real world contexts.
- There is lack of application of these expansions in new fields such as climate change analysis or biological sciences.

Where a large number of extensions based on Weibull distribution were presented or considered as a baseline, which can be viewed in the sources [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26].

This study highlights the importance of continuous expansion of Weibull distributions to keep pace with developments in data analysis. It also highlights how these expansions can address the shortcomings of traditional distribution, making them powerful tools modern statistical analysis.

2. SOME FAMILIES BASED ON WEIBULL DISTRIBUTION

2.1 Weibull-X family

This family was presented by Alzaatreh in 2014 [9], who relied on T-X method in methodology to find it [27]. This family has the CDF and PDF functions, respectively, as shown in the following form:

$$F(x,\alpha,\beta,\varepsilon) = 1 - e^{-\left\{-\log\left(\frac{1-F(x,\varepsilon)}{\alpha}\right)\right\}^{\beta}} \tag{5}$$

$$f(x,\alpha,\beta,\varepsilon) = \frac{\beta f(x,\varepsilon)}{\alpha\{1-F(x,\varepsilon)\}}\left\{-\log\left(\frac{1-F(x,\varepsilon)}{\alpha}\right)\right\}^{\beta-1} e^{-\left\{-\log\left(\frac{1-F(x,\varepsilon)}{\alpha}\right)\right\}^{\beta}} \tag{6}$$

Where $F(x,\varepsilon)$ and $f(x,\varepsilon)$ are CDF and PDF for any baseline distribution

For ease of reference, this family will be called Weibull-Type1(WT1). This family is characterized by flexibility in application, so to know flexibility and efficiency of WT1, the exponential distribution is replaced as a search direction for the family namely (WT1E) distribution as follows:

$$F_{WT1E}(x,\alpha,\beta,\lambda) = 1 - e^{-\left\{-\log\left(\frac{e^{-\lambda x}}{\alpha}\right)\right\}^{\beta}} \tag{7}$$

$$f_{WT1E}(x,\alpha,\beta,\lambda) = \frac{\beta\lambda}{\alpha}\left\{-\log\left(\frac{e^{-\lambda x}}{\alpha}\right)\right\}^{\beta-1} e^{-\left\{-\log\left(\frac{e^{-\lambda x}}{\alpha}\right)\right\}^{\beta}} \tag{8}$$

These function are shown in the following figures:

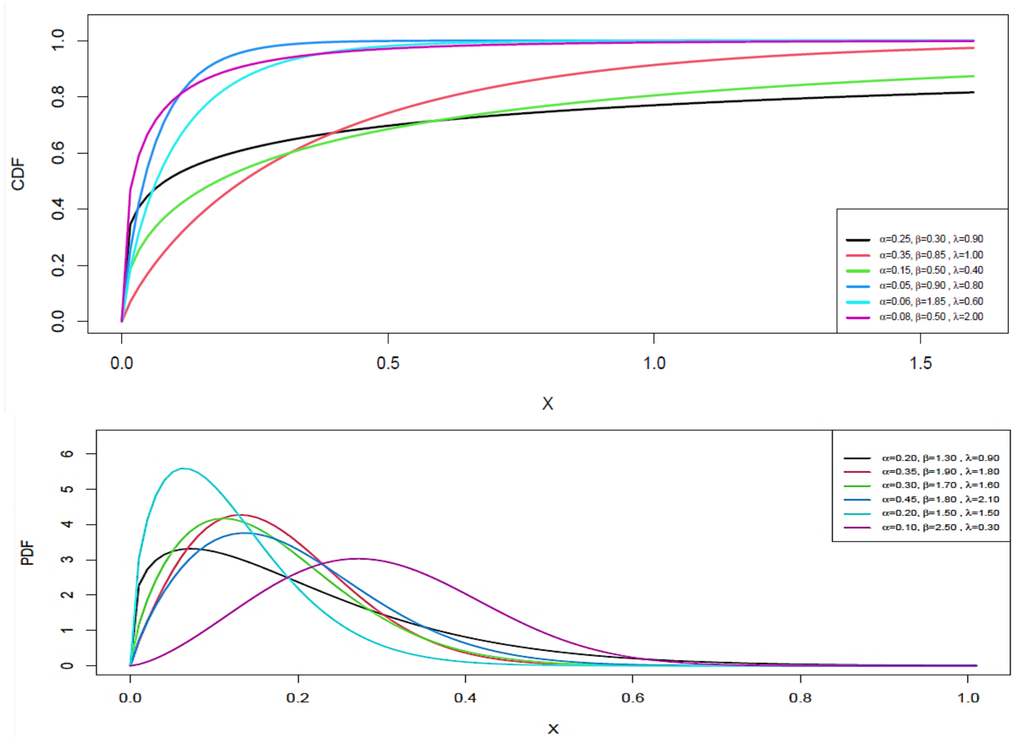


Fig..3. plot CDF and PDF functions for WT1E distribution with different values for α, β, and λ

2.2 Weibull-G family

This family was presented by Bourguignon et al. in 2014 [17], who relied on T-X method by same way in WT1 family but with different upper limitation for integral. This family has the CDF and PDF functions, respectively, as shown in the following form:

$$F(x, \alpha, \beta, \varepsilon) = 1 - e^{-\alpha\left\{\frac{F(x,\varepsilon)}{1-F(x,\varepsilon)}\right\}^{\beta}} \tag{1}$$

$$f(x, \alpha, \beta, \varepsilon) = \frac{\alpha\beta f(x,\varepsilon) F^{\beta-1}(x,\varepsilon)}{\{1 - F(x,\varepsilon)\}^{\beta+1}} e^{-\left\{-\log\left(\frac{1-F(x,\varepsilon)}{\alpha}\right)\right\}^{\beta}} \tag{2}$$

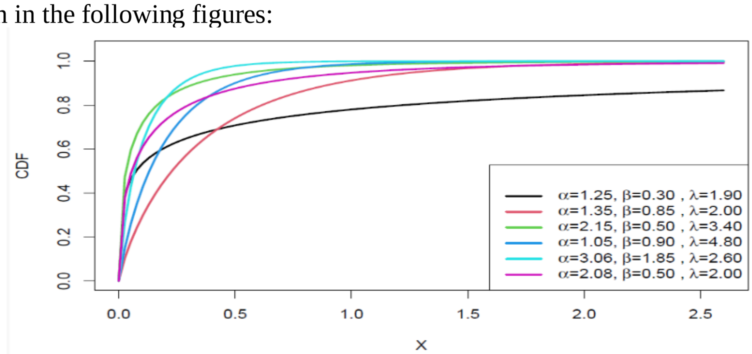
Where $F(x, \varepsilon)$ and $f(x, \varepsilon)$ are CDF and PDF for any baseline distribution

For ease of reference, this family will be called Weibull-Type2 (WT2) family. The main motivation for generating WT2 family in hope that they will yield better "t" in certain practical applications. Many statistical properties of WT2 have been studied, the exponential distribution is replaced as a search direction for the family namely (WT2E) distribution as follows:

$$F_{WT2E}(x, \alpha, \beta, \lambda) = 1 - e^{-\alpha\{e^{\lambda x}-1\}^{\beta}} \tag{11}$$

$$f_{WT2E}(x, \alpha, \beta, \lambda) = \frac{\alpha\beta\lambda(1 - e^{-\lambda x})^{\beta-1}}{e^{-\beta\lambda x}} e^{-\alpha\{e^{\lambda x}-1\}^{\beta}} \tag{12}$$

These function are shown in the following figures:



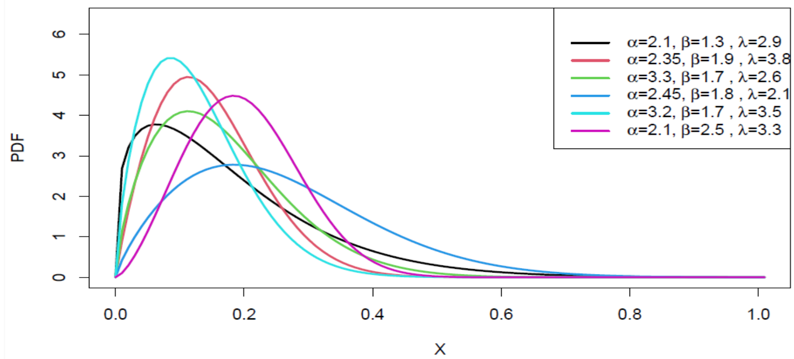


Fig.4. plot CDF and PDF functions for WT2E distribution with different values for α, β, and λ

2.3 Generalized Weibull (GW) family

This family was presented by Cordeiro et al. in 2015 [13], who relied on T-X method by same way in WT1 family but with upper limitation is $W(G) = -\log\big(1-F(x,\varepsilon)\big)$ for integral. This family has the CDF and PDF functions, respectively, as shown in the following form:

$$F(x,\alpha,\beta,\varepsilon) = 1 - e^{-\alpha\{-\log(1-F(x,\varepsilon))\}^{\beta}} \tag{13}$$

$$f(x,\alpha,\beta,\varepsilon) = \frac{\alpha\beta f(x,\varepsilon)}{1-F(x,\varepsilon)}\big\{-\log\big(1-F(x,\varepsilon)\big)\big\}^{\beta-1} e^{-\alpha\{-\log(1-F(x,\varepsilon))\}^{\beta}} \tag{14}$$

Where $F(x,\varepsilon)$ and $f(x,\varepsilon)$ are CDF and PDF for any baseline distribution

For ease of reference, this family will be called Weibull-Type3 (WT3) family. The main motivation for generating WT3 family is to provide a comprehensive treatment of general mathematical properties including quantum and generating functions, ordinary and incomplete moments, and other properties. The family also ensure that more continuous distribution are provided, the exponential distribution is replaced as a search direction for the family namely (WT3E) distribution as follows:

$$F_{WT3E}(x,\alpha,\beta,\lambda) = 1 - e^{-\alpha\{-\log(e^{-\lambda x})\}^{\beta}} \tag{15}$$

$$f_{WT3E}(x,\alpha,\beta,\lambda) = \alpha\beta\lambda\big\{-\log\big(e^{-\lambda x}\big)\big\}^{\beta-1} e^{-\alpha\{-\log(e^{-\lambda x})\}^{\beta}} \tag{16}$$

These function are shown in the following figures:

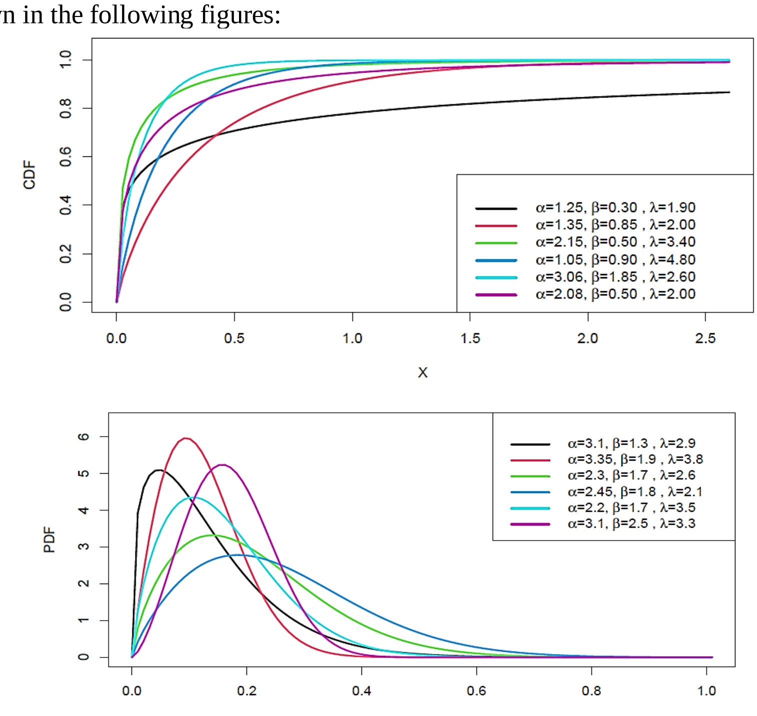


Fig..5. plot CDF and PDF functions for WT3E distribution with different values for α, β, and λ

2.4 New Weibull-G (NWG) family

This family was presented by Tahir et al. in 2015 [13], who relied on T-X method by same way in WT3 family but with upper limitation is $W(G) = -\log\big(F(x,\varepsilon)\big)$ for integral and modification of the basic function for the Weibull distribution. This family has the CDF and PDF functions, respectively, as shown in the following form:\

$$F(x,\alpha,\beta,\varepsilon) = e^{-\alpha\{-\log(F(x,\varepsilon))\}^{\beta}} \tag{17}$$

$$f(x,\alpha,\beta,\varepsilon) = \frac{\alpha\beta f(x,\varepsilon)}{F(x,\varepsilon)}\{-\log\big(F(x,\varepsilon)\big)\}^{\beta-1} e^{-\alpha\{-\log(F(x,\varepsilon))\}^{\beta}} \tag{18}$$

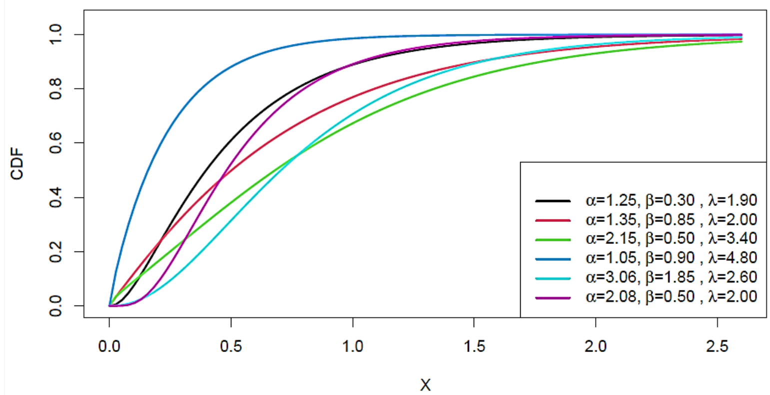
Where $F(x,\varepsilon)$ and $f(x,\varepsilon)$ are CDF and PDF for any baseline distribution

For ease of reference, this family will be called Weibull-Type4 (WT4) family. The family's density function is symmetrical, left-skewed, right-skewed, bathtub-shaped, or inverted J-shaped, and has increasing and decreasing risk rate, the exponential distribution is replaced as a search direction for the family namely (WT4E) distribution as follows:

$$F_{WT4E}(x,\alpha,\beta,\lambda) = 1 - e^{-\alpha\{-\log(1-e^{-\lambda x})\}^{\beta}} \tag{19}$$

$$f_{WT4E}(x,\alpha,\beta,\lambda) = \frac{\alpha\beta\lambda e^{-\lambda x}}{1-e^{-\lambda x}}\{-\log\big(1-e^{-\lambda x}\big)\}^{\beta-1} e^{-\alpha\{-\log(1-e^{-\lambda x})\}^{\beta}} \tag{20}$$

These function are shown in the following figures:



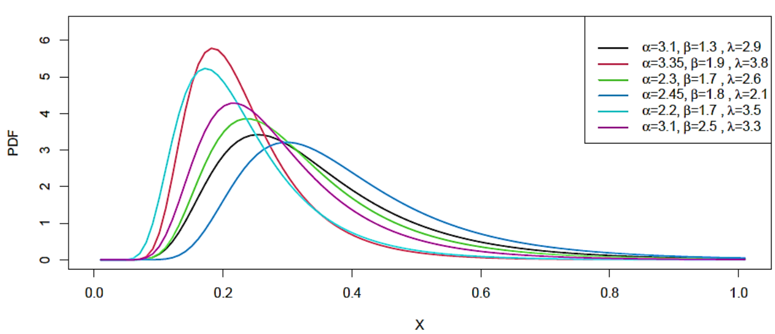


Fig..6. plot CDF and PDF functions for WT4E distribution with different values for α, β, and λ

2.5 New Weibull-G (NWG) family

This family was presented by BILAL et al. [10], who relied on T-X method by same upper limitation in WT3 family but and modification of parameter α. This family has the CDF and PDF functions, respectively, as shown in the following form:

$$F(x,\alpha,\beta,\varepsilon) = 1 - e^{-\alpha^{-\beta}\{-\log(1-F(x,\varepsilon))\}^{\beta}} \tag{21}$$

$$f(x,\alpha,\beta,\varepsilon) = \frac{\alpha^{-\beta}\beta f(x,\varepsilon)}{1-F(x,\varepsilon)}\{-\log\big(1-F(x,\varepsilon)\big)\}^{\beta-1} e^{-\alpha^{-\beta}\{-\log(1-F(x,\varepsilon))\}^{\beta}} \tag{22}$$

Where $F(x,\varepsilon)$ and $f(x,\varepsilon)$ are CDF and PDF for any baseline distribution

For ease of reference, this family will be called Weibull-Type5 (WT5) family. The introduction of WT5 is justified the compatibility of the newly developed class with its application in the field of quality control, the modified thing about the WT3 is addition of inverse power of β to the parameter α, which in turn more flexibility to WT5, the exponential distribution is replaced as a search direction for the family namely (WT6E) distribution as follows:

$$F_{WT5E}(x,\alpha,\beta,\lambda) = 1 - e^{-\alpha^{-\beta}\{-\log(e^{-\lambda x})\}^{\beta}} \quad (23)$$

$$f_{WT5E}(x,\alpha,\beta,\lambda) = \alpha^{-\beta}\beta\lambda\{-\log(e^{-\lambda x})\}^{\beta-1} e^{-\alpha^{-\beta}\{-\log(e^{-\lambda x})\}^{\beta}} \quad (24)$$

These function are shown in the following figures:

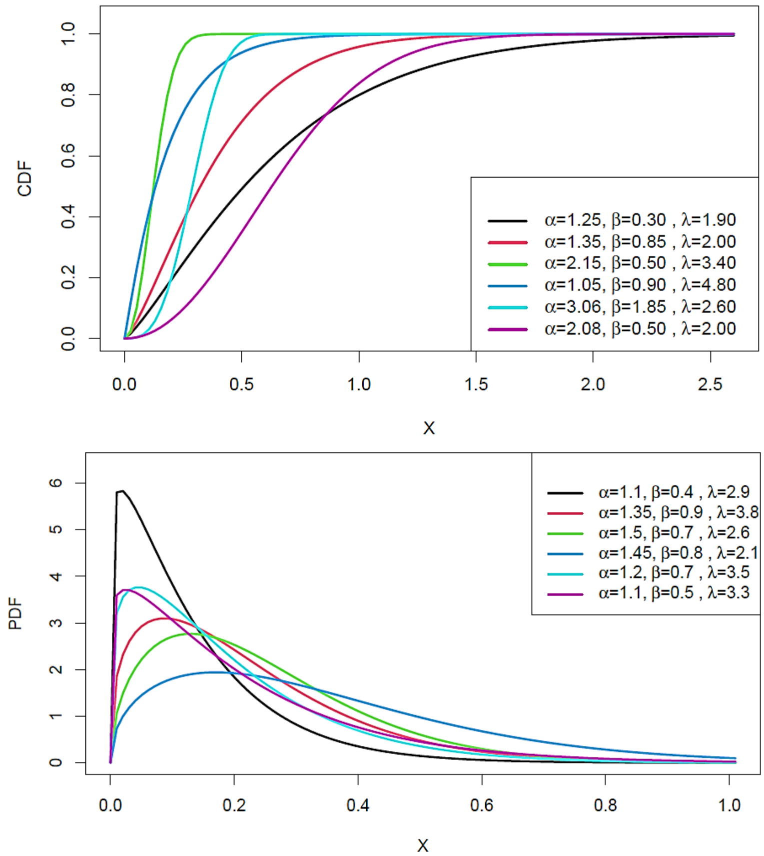


Fig.7. plot CDF and PDF functions for WT5E distribution with different values for α, β, and λ

2.6 Truncated Weibull-G(TWG) family

This family was presented by Najarzadegan et al. in 2017 [15], who relied on truncated Weibull-G family by Weibull inserting the truncated Weibull distribution with support [0,1] . This family has the CDF and PDF functions, respectively, as shown in the following form:

$$F(x,\alpha,\beta,\varepsilon) = \frac{1 - e^{-\alpha\{F(x,\varepsilon)\}^{\beta}}}{1 - e^{-\alpha}} \quad (25)$$

$$f(x,\alpha,\beta,\varepsilon) = \frac{\alpha.\beta.f(x,\varepsilon).\{F(x,\varepsilon)\}^{\beta-1}.e^{-\alpha\{F(x,\varepsilon)\}^{\beta}}}{1 - e^{-\alpha}} \quad (26)$$

Where $F(x,\varepsilon)$ and $f(x,\varepsilon)$ are CDF and PDF for any baseline distribution

For ease of reference, this family will be called Weibull-Type6 (WT6) family. The motivation for WT6 to introduce a new family of distribution as an alternative to Beta-G distribution with a flexible risk profile and greater reliability, the exponential distribution is replaced as a search direction for the family namely (WT6E) distribution as follows:

$$F_{WT6E}(x,\alpha,\beta,\lambda) = \frac{1 - e^{-\alpha\{1-e^{-\lambda x}\}^{\beta}}}{1 - e^{-\alpha}} \quad (27)$$

$$f_{WT6E}(x,\alpha,\beta,\lambda) = \frac{\alpha.\beta.\lambda e^{-\lambda x}.\{1 - e^{-\lambda x}\}^{\beta-1}.e^{-\alpha\{1-e^{-\lambda x}\}^{\beta}}}{1 - e^{-\alpha}} \quad (28)$$

These function are shown in the following figures:

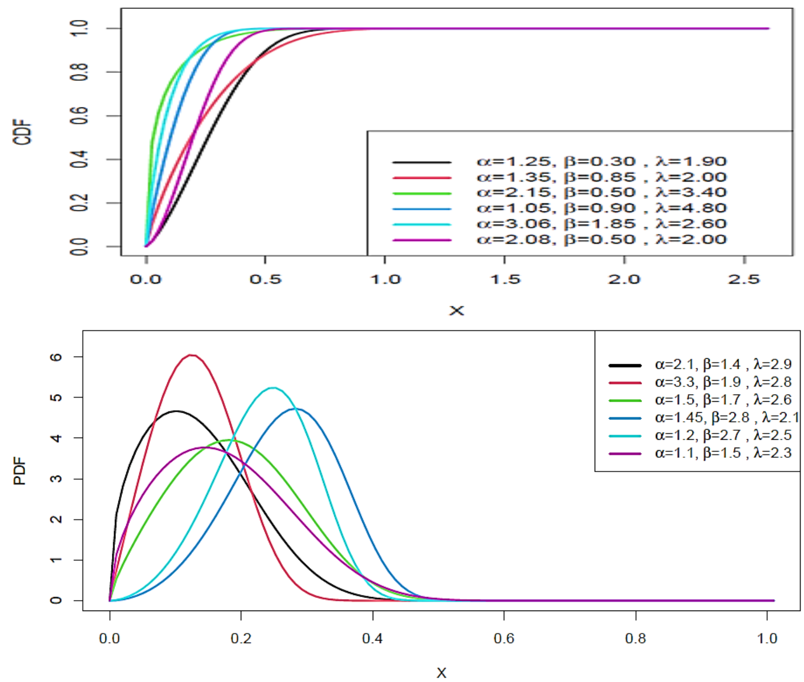


Fig.8. plot CDF and PDF functions for WT6E distribution with different values for α, β, and λ

2.7 Inverse Weibull-G (IWG) family

This family was presented by Hassan and Nassr in 2018 [8], who relied on Inverse Weibull distribution, T-X method and odd function for upper limitation for integral $W(G) = \frac{F(x,\varepsilon)}{1-F(x,\varepsilon)}$. This family has the CDF and PDF functions, respectively, as shown in the following form:

$$F(x, \alpha, \beta, \varepsilon) = e^{-\alpha^{\beta}\left\{\frac{F(x,\varepsilon)}{1-F(x,\varepsilon)}\right\}^{-\beta}} \tag{29}$$

$$f(x, \alpha, \beta, \varepsilon) = \frac{\alpha^{\beta}.\beta.f(x,\varepsilon).\{F(x,\varepsilon)\}^{-\beta-1}}{\{1-F(x,\varepsilon)\}^{-\beta+1}} e^{-\alpha^{\beta}\left\{\frac{F(x,\varepsilon)}{1-F(x,\varepsilon)}\right\}^{-\beta}} \tag{30}$$

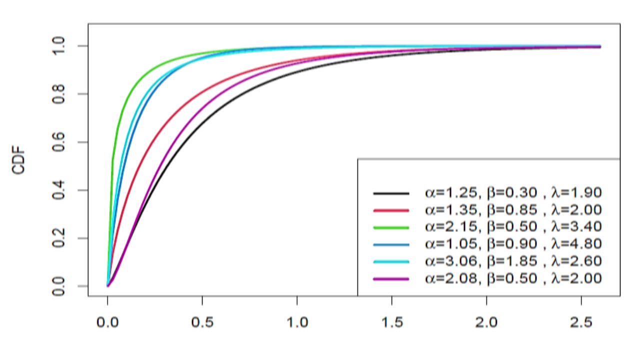
Where $F(x,\varepsilon)$ and $f(x,\varepsilon)$ are CDF and PDF for any baseline distribution

For ease of reference, this family will be called Weibull-Type7 (WT7) family. The motivation for WT7 with that it will attract a wider application in some area, the exponential distribution is replaced as a search direction for the family namely (WT7E) distribution as follows:

$$F_{WT7E}(x, \alpha, \beta, \lambda) = e^{-\alpha^{\beta}\left\{\frac{1-e^{-\lambda x}}{e^{-\lambda x}}\right\}^{-\beta}} \tag{31}$$

$$f_{WT7E}(x, \alpha, \beta, \lambda) = \frac{\alpha^{\beta}.\beta.\lambda.\{1-e^{-\lambda x}\}^{-\beta-1}}{e^{\beta\lambda x}} e^{-\alpha^{\beta}\left\{\frac{1-e^{-\lambda x}}{e^{-\lambda x}}\right\}^{-\beta}} \tag{32}$$

These function are shown in the following figures:



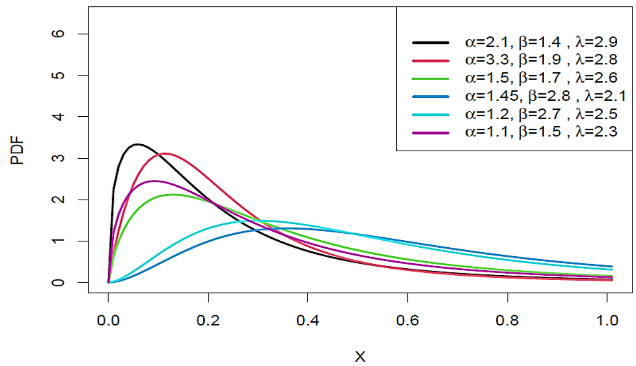


FIGURE.9 plot CDF and PDF functions for WT7E distribution with different values for α, β, and λ

2.8 Extended odd Weibull-G (ExOW) family

This family was presented by Alizadah et al. in 2019 [19], who relied on Weibull distribution, T-X method and odd function for upper limitation for integral $W(G) = \frac{F(x,\varepsilon)}{1-F(x,\varepsilon)}$. This family has the CDF and PDF functions, respectively, as shown in the following form:

$$F(x,\alpha,\beta,\varepsilon) = 1 - \left\{1 + \beta\left\{\frac{F(x,\varepsilon)}{1-F(x,\varepsilon)}\right\}^{\alpha}\right\}^{-\frac{1}{\beta}} \tag{33}$$

$$f(x,\alpha,\beta,\varepsilon) = \frac{\alpha f(x,\varepsilon).\{F(x,\varepsilon)\}^{\alpha-1}}{\{1-F(x,\varepsilon)\}^{\alpha+1}}\left\{1 + \beta\left\{\frac{F(x,\varepsilon)}{1-F(x,\varepsilon)}\right\}^{\alpha}\right\}^{-\frac{1}{\beta}-1} \tag{34}$$

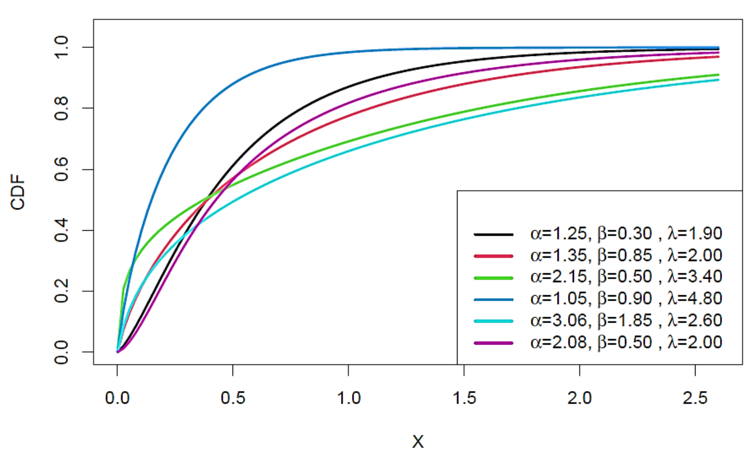
Where $F(x,\varepsilon)$ and $f(x,\varepsilon)$ are CDF and PDF for any baseline distribution

For ease of reference, this family will be called Weibull-Type8 (WT8) family. The motivation for WT8 is that the generated method will benefit from the flexibility of the underling distribution for data modeling and it's also motivated by its ability to model data with increasing and decreasing failure rates, single model and bimodal, the exponential distribution is replaced as a search direction for the family namely (WT8E) distribution as follows:

$$F_{WT8E}(x,\alpha,\beta,\lambda) = 1 - \left\{1 + \beta\left\{\frac{1-e^{-\lambda x}}{e^{-\lambda x}}\right\}^{\alpha}\right\}^{-\frac{1}{\beta}} \tag{35}$$

$$f_{WT8E}(x,\alpha,\beta,\lambda) = \frac{\alpha\lambda.\{1-e^{-\lambda x}\}^{\alpha-1}}{e^{-\alpha\lambda x}}\left\{1 + \beta\left\{\frac{1-e^{-\lambda x}}{e^{-\lambda x}}\right\}^{\alpha}\right\}^{-\frac{1}{\beta}-1} \tag{36}$$

These function are shown in the following figures:



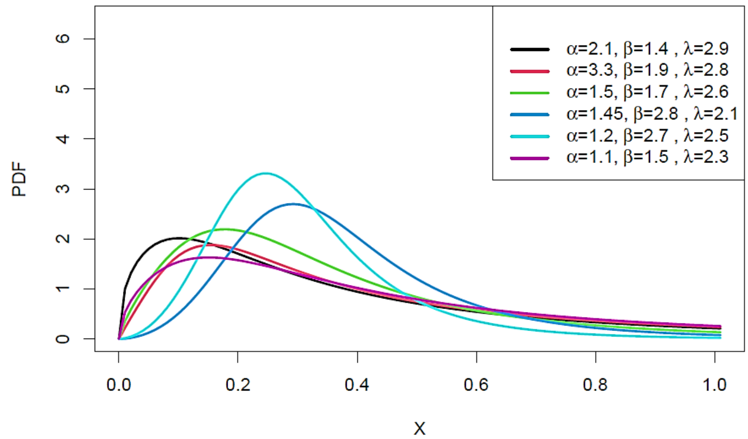


Fig.10. plot CDF and PDF functions for WT8E distribution with different values for α, β, and λ

2.9 Nasir Weibull Generalized family (NW-G) family

This family was presented by Farrukh and Nasir in 2019 [28], who relied on Weibull distribution, T-X method and odd function for upper limitation for integral $W(G) = \frac{-\log F(x,\varepsilon)}{F(x,\varepsilon)}$. This family has the CDF and PDF functions, respectively, as shown in the following form:

$$F(x,\alpha,\beta,\varepsilon) = 1 - e^{-\alpha\left\{\frac{-\log F(x,\varepsilon)}{F(x,\varepsilon)}\right\}^{\beta}} \tag{37}$$

$$f(x,\alpha,\beta,\varepsilon) = \frac{\alpha\beta f(x,\varepsilon).\{-\log F(x,\varepsilon)\}^{\beta-1}}{\{F(x,\varepsilon)\}^{\beta+1}} e^{-\alpha\left\{\frac{-\log F(x,\varepsilon)}{F(x,\varepsilon)}\right\}^{\beta-1}} [1 - \log F(x,\varepsilon)] \tag{38}$$

Where $F(x,\varepsilon)$ and $f(x,\varepsilon)$ are CDF and PDF for any baseline distribution

For ease of reference, this family will be called Weibull-Type9 (WT9) family. The motivation for WT9 to more flexibility to modeling data, the exponential distribution is replaced as a search direction for the family namely (WT9E) distribution as follows:

$$F_{WT9E}(x,\alpha,\beta,\lambda) = 1 - e^{-\alpha\left\{\frac{-\log[1-e^{-\lambda x}]}{1-e^{-\lambda x}}\right\}^{\beta}} \tag{39}$$

$$f_{WT9E}(x,\alpha,\beta,\lambda) = \frac{\alpha\beta\lambda e^{-\lambda x}.\{-\log[1-e^{-\lambda x}]\}^{\beta-1}}{\{1-e^{-\lambda x}\}^{\beta+1}} e^{-\alpha\left\{\frac{-\log[1-e^{-\lambda x}]}{1-e^{-\lambda x}}\right\}^{\beta-1}} \left[1 - \log[1-e^{-\lambda x}]\right] \tag{40}$$

These function are shown in the following figures:

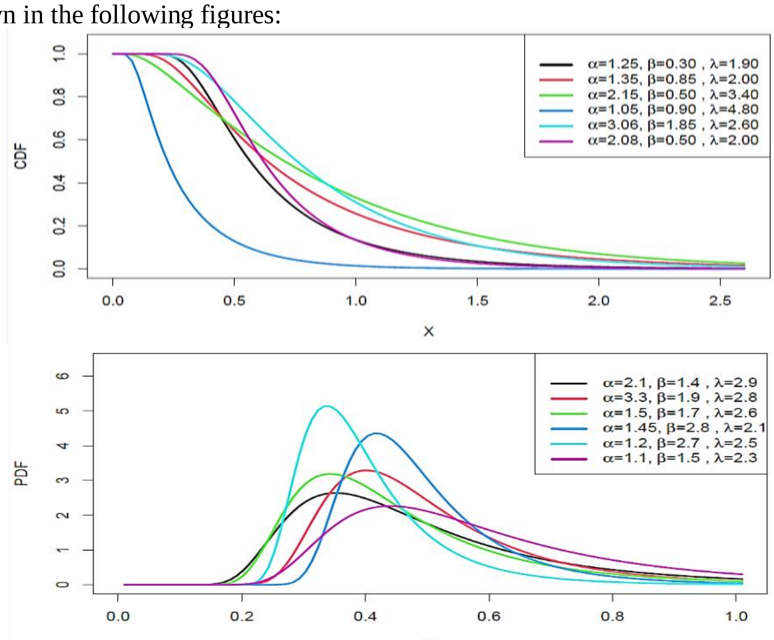


Fig.11. plot CDF and PDF functions for WT9E distribution with different values for α, β, and λ

2.10 Exponentiated Truncated Inverse Weibull Generated family (ETIW-G) family

This family was presented by M. Almarashi et al. in 2020 [5], who relied on inverse Weibull distribution, and truncated method. This family has the CDF and PDF functions, respectively, as shown in the following form:

$$F(x,\alpha,\beta,\varepsilon)=\left(1-e^{1-\{1-F(x,\varepsilon)\}^{-\beta}}\right)^{\alpha} \tag{41}$$

$$f(x,\alpha,\beta,\varepsilon)=\alpha\beta f(x,\varepsilon).\{1-F(x,\varepsilon)\}^{-\beta-1}e^{1-\{1-F(x,\varepsilon)\}^{-\beta}}\left(1-e^{1-\{1-F(x,\varepsilon)\}^{-\beta}}\right)^{\alpha-1} \tag{42}$$

Where $F(x,\varepsilon)$ and $f(x,\varepsilon)$ are CDF and PDF for any baseline distribution

For ease of reference, this family will be called Weibull-Type10 (WT10) family. The motivation for create WT10 using inverse distribution or truncated distribution as key generators is to exploit some of their specific properties to create original and flexible distributions, such as their simplicity of use, their economy of transactions, their inverted bathtub risk ratio properties, and their ability to produce heavy tails, the exponential distribution is replaced as a search direction for the family namely (WT10E) distribution as follows:

$$F_{WT10E}(x,\alpha,\beta,\lambda)=\left(1-e^{1-\{e^{-\lambda x}\}^{-\beta}}\right)^{\alpha} \tag{43}$$

$$f_{WT10E}(x,\alpha,\beta,\lambda)=\alpha\beta\lambda e^{-\lambda x}.\{e^{-\lambda x}\}^{-\beta-1}e^{1-\{e^{-\lambda x}\}^{-\beta}}\left(1-e^{1-\{e^{-\lambda x}\}^{-\beta}}\right)^{\alpha-1} \tag{44}$$

These function are shown in the following figures:

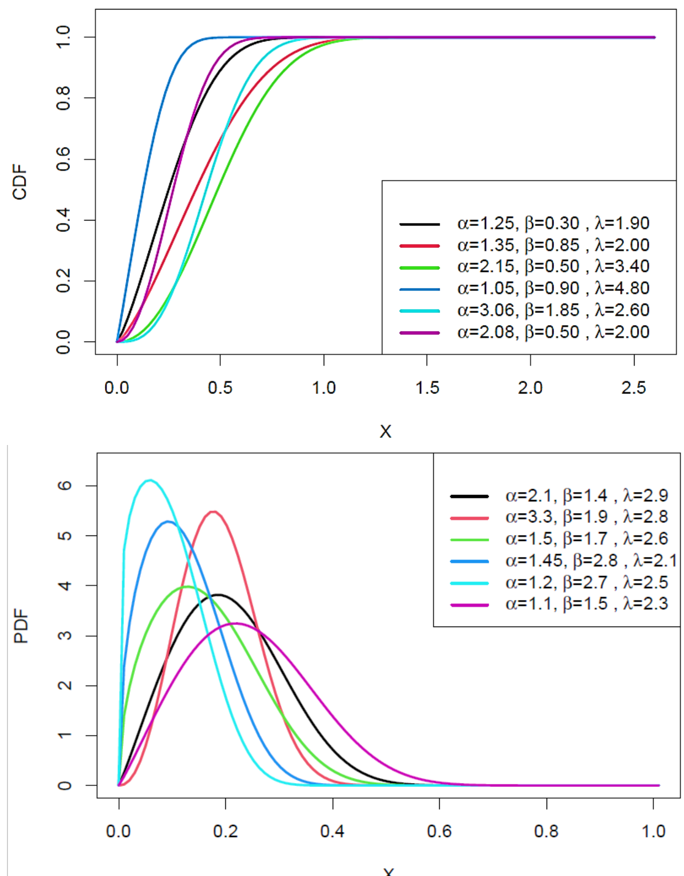


Fig.12. plot CDF and PDF functions for WT10E distribution with different values for α,β, and λ

2.11 Odd Inverse Weibull Generated family (IW-G) family

This family was presented by Yassmen et al. in 2024 [25], who relied on inverse Weibull distribution, and T-X method with upper limitation for integral $W(G)=\frac{F(x,\varepsilon)^{\alpha}}{[1-F(x,\varepsilon)]^{\alpha}}$. This family has the CDF and PDF functions, respectively, as shown in the following form:

$$F(x,\alpha,\beta,\varepsilon)=e^{-\left\{\frac{F(x,\varepsilon)^{\alpha}}{[1-F(x,\varepsilon)]^{\alpha}}\right\}^{-\beta}} \tag{45}$$

$$f(x,\alpha,\beta,\varepsilon)=\frac{\alpha\beta f(x,\varepsilon).\{[1-F(x,\varepsilon)]^{\alpha}\}^{\beta-1}}{\{F(x,\varepsilon)\}^{\alpha\beta+1}}e^{-\left\{\frac{F(x,\varepsilon)^{\alpha}}{[1-F(x,\varepsilon)]^{\alpha}}\right\}^{-\beta}} \quad (46)$$

Where $F(x,\varepsilon)$ and $f(x,\varepsilon)$ are CDF and PDF for any baseline distribution

For ease of reference, this family will be called Weibull-Type11 (WT11) family. The WT11 is considered a generalization of WT7 if a new integral term is introduced by raising the singularity function to a power, which seems to give more flexibility than WT7 family, the exponential distribution is replaced as a search direction for the family namely (WT11E) distribution as follows:

$$F_{WT11E}(x,\alpha,\beta,\lambda)=e^{-\left\{\frac{(1-e^{-\lambda x})^{\alpha}}{[e^{-\lambda x}]^{\alpha}}\right\}^{-\beta}} \quad (47)$$

$$f_{WT11E}(x,\alpha,\beta,\lambda)=\frac{\alpha\beta\lambda e^{-\lambda x}.\left\{\left[e^{-\lambda x}\right]^{\alpha}\right\}^{\beta-1}}{\{1-e^{-\lambda x}\}^{\alpha\beta+1}}e^{-\left\{\frac{(1-e^{-\lambda x})^{\alpha}}{[e^{-\lambda x}]^{\alpha}}\right\}^{-\beta}} \quad (48)$$

These function are shown in the following figures:

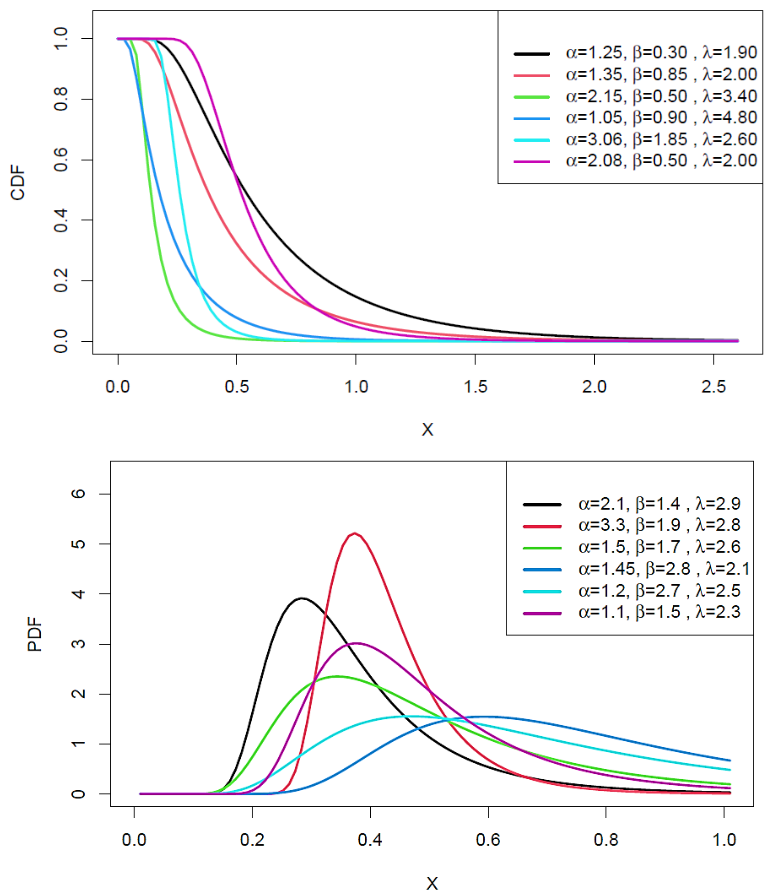


Fig.13. plot CDF and PDF functions for WT11E distribution with different values for α,β, and λ

2.12 Hybrid Weibull-G (HWG) family

This family was presented by Noori and khaleel in 2024 [21], who relied on Weibull distribution, and T-X method with upper limitation for integral $W(G)=-F(x,\varepsilon)log\,[1-F(x,\varepsilon)$. This family has the CDF and PDF functions, respectively, as shown in the following form:

$$F(x,\alpha,\beta,\varepsilon)=1-e^{-\alpha[-F(x,\varepsilon)\log[1-F(x,\varepsilon)]]^{\beta}} \quad (49)$$

$$f(x,\alpha,\beta,\varepsilon)=\alpha\beta\, f(x,\varepsilon)\left[\frac{F(x,\varepsilon)}{1-F(x,\varepsilon)}-\log(1-F(x,\varepsilon))\right]\times[-F(x,\varepsilon)log\,[1-F(x,\varepsilon)]]^{\beta-1}e^{-\alpha[-F(x,\varepsilon)log\,[1-F(x,\varepsilon)]]^{\beta}} \quad (50)$$

Where $F(x,\varepsilon)$ and $f(x,\varepsilon)$ are CDF and PDF for any baseline distribution

For ease of reference, this family will be called Weibull-Type12 (WT12) family. The WT12, by virtue of its integration of two limits of the formation of continuous families, combine the strength of these function to add more flexibility in practical applications, the exponential distribution is replaced as a search direction for the family namely (WT12E) distribution as follows:

$$F_{WT12E}(x,\alpha,\beta,\lambda) = 1 - e^{-\alpha[-(1-e^{-\lambda x})\log[e^{-\lambda x}]]^{\beta}} \quad (51)$$

$$f_{WT12E}(x,\alpha,\beta,\lambda) = \alpha\beta\,\lambda e^{-\lambda x}\left[\frac{1-e^{-\lambda x}}{e^{-\lambda x}} - \log(e^{-\lambda x})\right] \times \left[-(1-e^{-\lambda x})log\,[e^{-\lambda x}]\right]^{\beta-1} e^{-\alpha[-(1-e^{-\lambda x})log\,[e^{-\lambda x}]]^{\beta}} \quad (52)$$

These function are shown in the following figures:

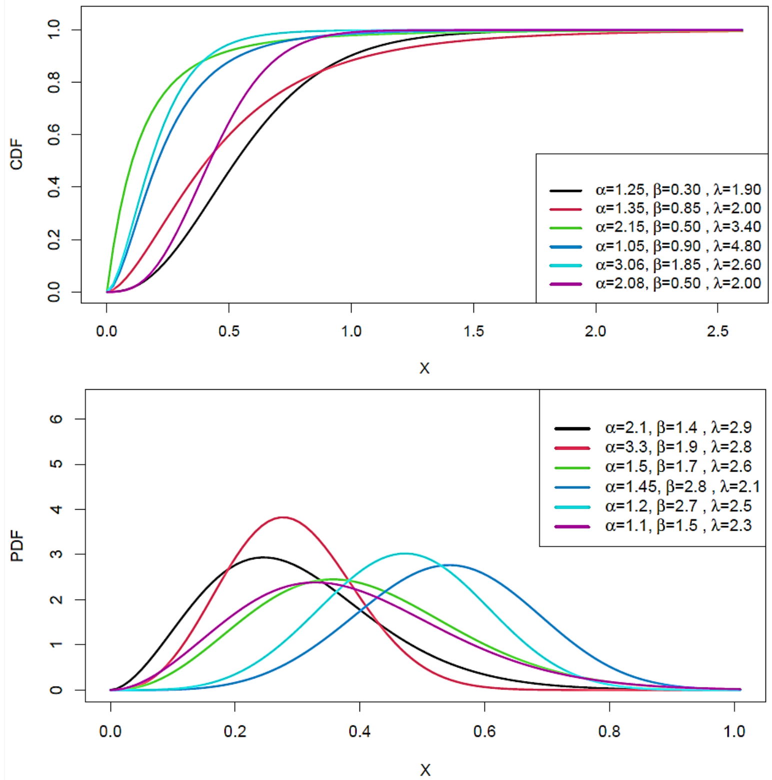


Fig.14. plot CDF and PDF functions for WT12E distribution with different values for α, β, and λ

3. SIMULATION

To determine the efficiency of each of the models (WT1E,…,WT12E) which was presented in previous section, Monte Carlo simulation were conducted for three methods: Maximal likelihood estimation (MLE) [29], Least squares estimation (LSE) [30], and Weighted Least squares estimation (WLSE) [31]. The generated sample sizes were based on n=50, 100, 150 and 200 to 1000. To evaluate the performance, the mean square error (MSE) [22], root mean square error (RMSE) [32], and bias in the estimated parameters were calculated [33],

The algorithm for applied a Monte Carlo simulation to determine the best expansion based on statistical accuracy criteria:

1. Choose the statistical model distribution, and determine the actual parameters to be estimated.
2. List the different estimation technique.
3. Select multiple sample sizes (n=50, 100, 150 and 200) to test the effect of sample size. Then, generate N iteration (e.g., 1000 samples) using a random number generator based on chosen distribution.
4. Apply the estimation technique to each sample. Parameters estimates are calculate using all the selected estimation techniques.
5. Compute the statistical accuracy criteria for each technique across all iterations, and then the values are compared between techniques to determine the best performer .
6. Rank techniques based on their performance according to three statistical criteria. Then, identify the technique with the lowest values as a measure of performance advantage

The results shown in tables 1 to 12 respectively.

TABLE I. MONTE CARLO SIMULATIONS FOR THE WTIE

$\alpha = 2.4, \quad \beta = 3.5, \quad \lambda = 1.7$									
N	**Est.**	**Ess. Par.**	**MLE**	**LSE**	**WLSE**	**N**	**MLE**	**LSE**	**WLSE**
50	Mean	$\hat{\alpha}$	2.46011	2.44792	2.44912	100	2.45719	2.449145	2.443552
		$\hat{\beta}$	3.60296	3.49601	3.52150		3.54789	3.492562	3.51549
		$\hat{\lambda}$	1.74359	1.72983	1.73272		1.74194	1.734413	1.731481
	MSE	$\hat{\alpha}$	0.01403	0.01629	0.01452		0.006765	0.008018	0.006451
		$\hat{\beta}$	0.17135	0.23067	0.19532		0.07974	0.117829	0.09599
		$\hat{\lambda}$	0.00896	0.01014	0.00933		0.00469	0.005085	0.004224
	RMSE	$\hat{\alpha}$	0.11845	0.12766	0.12050		0.08225	0.089546	0.080324
		$\hat{\beta}$	0.41394	0.48028	0.44196		0.28239	0.343263	0.30982
		$\hat{\lambda}$	0.09465	0.10070	0.09659		0.06854	0.071310	0.064998
	Bias	$\hat{\alpha}$	0.06011	0.04792	0.04912		0.057196	0.049145	0.043552
		$\hat{\beta}$	0.10296	0.00398	0.02150		0.04789	0.007437	0.01549
		$\hat{\lambda}$	0.04359	0.02983	0.03272		0.04194	0.034413	0.031481
N	**Est.**		MLE	LSE	WLSE	N	MLE	LSE	WLSE
150	Mean	$\hat{\alpha}$	2.45802	2.44342	2.44777	200	2.455070	2.441467	2.446793
		$\hat{\beta}$	3.53065	3.49834	3.51234		3.508713	3.477412	3.490347
		$\hat{\lambda}$	1.74083	1.72913	1.73276		1.742833	1.731428	1.736122
	MSE	$\hat{\alpha}$	0.005892	0.005783	0.00496		0.004779	0.003679	0.003979
		$\hat{\beta}$	0.053418	0.07779	0.063777		0.037458	0.051184	0.041774
		$\hat{\lambda}$	0.003676	0.003754	0.003285		0.003284	0.002519	0.002883
	RMSE	$\hat{\alpha}$	0.076765	0.076048	0.070433		0.069135	0.060661	0.063082
		$\hat{\beta}$	0.231125	0.27891	0.25254		0.193543	0.226240	0.204387
		$\hat{\lambda}$	0.060631	0.061277	0.057316		0.057310	0.050193	0.053696
	Bias	$\hat{\alpha}$	0.058026	0.043429	0.047779		0.055070	0.041467	0.046793
		$\hat{\beta}$	0.030653	0.001655	0.012341		0.008713	0.02258	0.009652
		$\hat{\lambda}$	0.040832	0.029133	0.032762		0.042833	0.031428	0.036122

TABLE II. MONTE CARLO SIMULATIONS FOR THE WT2E

$\alpha = 2.4, \quad \beta = 3.5, \quad \lambda = 1.7$									
N	**Est.**	**Ess. Par.**	**MLE**	**LSE**	**WLSE**	**N**	**MLE**	**LSE**	**WLSE**
50	Mean	$\hat{\alpha}$	2.50843	2.49461	2.48295	100	2.524882	2.492312	2.496614
		$\hat{\beta}$	3.58035	3.459142	3.49251		3.554078	3.507087	3.527307
		$\hat{\lambda}$	1.688911	1.67488	1.68367		1.682361	1.683105	1.684941
	MSE	$\hat{\alpha}$	0.049799	0.11247	0.091375		0.035290	0.067612	0.051490
		$\hat{\beta}$	0.15483	0.215610	0.177541		0.082108	0.118400	0.098640
		$\hat{\lambda}$	0.007211	0.013859	0.012078		0.0036022	0.006367	0.004887
	RMSE	$\hat{\alpha}$	0.22315	0.33537	0.30228		0.187856	0.260024	0.226915
		$\hat{\beta}$	0.393489	0.46433	0.421357		0.286545	0.3440932	0.314070
		$\hat{\lambda}$	0.084922	0.117727	0.109901		0.0600186	0.07979	0.0699089
	Bias	$\hat{\alpha}$	0.108438	0.094614	0.082953		0.124882	0.092312	0.096614
		$\hat{\beta}$	0.08035	0.040857	0.007481		0.054078	0.007087	0.027307
		$\hat{\lambda}$	0.011088	0.025116	0.016329		0.0176381	0.016894	0.015058
N	**Est.**		MLE	LSE	WLSE	N	MLE	LSE	WLSE
150	Mean	$\hat{\alpha}$	2.523791	2.51318	2.50248	200	2.520207	2.499802	2.500697
		$\hat{\beta}$	3.533923	3.502811	3.51633		3.531456	3.51035	3.519757
		$\hat{\lambda}$	1.678408	1.676117	1.680372		1.6785363	1.6793800	1.680796
	MSE	$\hat{\alpha}$	0.029432	0.060633	0.045075		0.024416	0.047830	0.03709
		$\hat{\beta}$	0.050548	0.074878	0.060896		0.040650	0.05675	0.04734
		$\hat{\lambda}$	0.002773	0.004581	0.003363		0.001874	0.002872	0.002309
	RMSE	$\hat{\alpha}$	0.171557	0.24623	0.21231		0.156258	0.218701	0.192593
		$\hat{\beta}$	0.22483	0.273638	0.24677		0.201619	0.238240	0.217593
		$\hat{\lambda}$	0.052663	0.067688	0.057998		0.043291	0.053592	0.0480545
	Bias	$\hat{\alpha}$	0.123791	0.113181	0.10248		0.120207	0.099802	0.100697
		$\hat{\beta}$	0.033923	0.002811	0.016335		0.031456	0.010352	0.019757
		$\hat{\lambda}$	0.021591	0.023882	0.019627		0.0214636	0.0206199	0.0192033

TABLE III. MONTE CARLO SIMULATIONS FOR THE WT3E

$\alpha = 2.4$, $\beta = 3.5$, $\lambda = 1.7$									
N	**Est.**	**Ess. Par.**	**MLE**	**LSE**	**WLSE**	**N**	**MLE**	**LSE**	**WLSE**
50	Mean	$\hat{\alpha}$	2.475599	2.51420	2.50999	100	2.441321	2.452803	2.447863
		$\hat{\beta}$	3.79394	3.63379	3.65450		3.66554	3.56362	3.594077
		$\hat{\lambda}$	1.749902	1.683292	1.699722		1.730050	1.696211	1.710845
	MSE	$\hat{\alpha}$	0.163272	0.238326	0.25206		0.074457	0.105603	0.099695
		$\hat{\beta}$	1.15334	1.62944	1.31499		0.46465	0.632601	0.508314
		$\hat{\lambda}$	0.211703	0.249320	0.236307		0.094006	0.110949	0.119210
	RMSE	$\hat{\alpha}$	0.404069	0.488186	0.50205		0.272869	0.324967	0.315745
		$\hat{\beta}$	1.073940	1.276495	1.146734		0.681653	0.795362	0.712962
		$\hat{\lambda}$	0.460112	0.499320	0.486114		0.306605	0.333090	0.345268
	Bias	$\hat{\alpha}$	0.075599	0.114202	0.109999		0.041321	0.052803	0.047863
		$\hat{\beta}$	0.293948	0.133794	0.15450		0.16554	0.063624	0.094077
		$\hat{\lambda}$	0.049902	0.016707	0.000277		0.030050	0.0037884	0.010845
N	**Est.**		MLE	LSE	WLSE	N	MLE	LSE	WLSE
150	Mean	$\hat{\alpha}$	2.456008	2.46565	2.448453	200	2.446910	2.457797	2.456373
		$\hat{\beta}$	3.60428	3.540402	3.561646		3.576044	3.524846	3.544304
		$\hat{\lambda}$	1.698107	1.678119	1.690860		1.688718	1.670582	1.673141
	MSE	$\hat{\alpha}$	0.047032	0.079099	0.051348		0.032393	0.067221	0.045531
		$\hat{\beta}$	0.26687	0.400179	0.318741		0.200118	0.290980	0.229266
		$\hat{\lambda}$	0.056207	0.077619	0.057426		0.040159	0.049309	0.035642
	RMSE	$\hat{\alpha}$	0.216870	0.281246	0.226601		0.179980	0.259271	0.213381
		$\hat{\beta}$	0.51660	0.632597	0.564572		0.447345	0.539426	0.478817
		$\hat{\lambda}$	0.237080	0.278602	0.239638		0.200399	0.222058	0.188792
	Bias	$\hat{\alpha}$	0.056008	0.065650	0.048453		0.046910	0.057797	0.056373
		$\hat{\beta}$	0.104286	0.040402	0.061646		0.076044	0.024846	0.044304
		$\hat{\lambda}$	0.001892	0.021880	0.009139		0.011281	0.029417	0.026858

TABLE IV. MONTE CARLO SIMULATIONS FOR THE WT4E

$\alpha = 2.4$, $\beta = 3.5$, $\lambda = 1.7$									
N	**Est.**	**Ess. Par.**	**MLE**	**LSE**	**WLSE**	**N**	**MLE**	**LSE**	**WLSE**
50	Mean	$\hat{\alpha}$	2.466868	2.384677	2.407680	100	2.434884	2.3983360	2.411491
		$\hat{\beta}$	3.595104	3.605006	3.597582		3.594910	3.588923	3.589182
		$\hat{\lambda}$	1.747088	1.744442	1.743636		1.7470359	1.7413005	1.7423942
	MSE	$\hat{\alpha}$	0.083771	0.109081	0.092288		0.042393	0.0579207	0.049078
		$\hat{\beta}$	0.025645	0.033218	0.027665		0.016626	0.017852	0.016583
		$\hat{\lambda}$	0.010630	0.010857	0.010396		0.0069918	0.0064195	0.0062093
	RMSE	$\hat{\alpha}$	0.289433	0.330274	0.303790		0.205896	0.2406673	0.221536
		$\hat{\beta}$	0.160142	0.182260	0.166328		0.128942	0.133613	0.1287758
		$\hat{\lambda}$	0.103102	0.104199	0.101962		0.0836170	0.0801219	0.0787991
	Bias	$\hat{\alpha}$	0.066868	0.015322	0.007680		0.034884	0.0016639	0.011491
		$\hat{\beta}$	0.095104	0.105006	0.097582		0.094910	0.088923	0.089182
		$\hat{\lambda}$	0.047088	0.044442	0.043636		0.0470359	0.0413005	0.0423942
N	**Est.**		MLE	LSE	WLSE	N	MLE	LSE	WLSE
150	Mean	$\hat{\alpha}$	2.435869	2.415422	2.423689	200	2.420547	2.4085510	2.412654
		$\hat{\beta}$	3.586388	3.579744	3.576966		3.587847	3.578616	3.580756
		$\hat{\lambda}$	1.744332	1.740082	1.739339		1.7433171	1.7366070	1.7384875
	MSE	$\hat{\alpha}$	0.025920	0.036903	0.030203		0.018904	0.0281324	0.022670
		$\hat{\beta}$	0.012484	0.012887	0.011181		0.011684	0.010898	0.010753
		$\hat{\lambda}$	0.004968	0.004748	0.004277		0.0041827	0.0036094	0.0036977
	RMSE	$\hat{\alpha}$	0.160998	0.192101	0.173791		0.1374943	0.1677273	0.150567
		$\hat{\beta}$	0.111735	0.113521	0.105743		0.108094	0.1043967	0.1037006
		$\hat{\lambda}$	0.070486	0.068907	0.065406		0.0646738	0.0600783	0.0608088
	Bias	$\hat{\alpha}$	0.035869	0.015422	0.023689		0.0205477	0.0085510	0.0126548
		$\hat{\beta}$	0.086388	0.079744	0.076966		0.087847	0.0786166	0.0807563
		$\hat{\lambda}$	0.044332	0.040082	0.039339		0.0433171	0.0366070	0.0384875

TABLE V. MONTE CARLO SIMULATIONS FOR THE WT5E

$\alpha = 2.4$, $\beta = 3.5$, $\lambda = 1.7$									
N	**Est.**	**Ess. Par.**	**MLE**	**LSE**	**WLSE**	**N**	**MLE**	**LSE**	**WLSE**
50	Mean	$\hat{\alpha}$	2.461023	2.443411	2.443550	100	2.4549728	2.441562	2.4474951
		$\hat{\beta}$	3.573021	3.494195	3.517642		3.539111	3.4982267	3.514487
		$\hat{\lambda}$	1.747069	1.731565	1.732979		1.7440804	1.7330376	1.7378170
	MSE	$\hat{\alpha}$	0.010835	0.020051	0.013745		0.0061696	0.006099	0.0063319
		$\hat{\beta}$	0.145244	0.201701	0.169355		0.068994	0.0927433	0.078097
		$\hat{\lambda}$	0.008069	0.012800	0.009373		0.0042202	0.0044963	0.0044036
	RMSE	$\hat{\alpha}$	0.104094	0.141601	0.117243		0.0785473	0.078099	0.0795735
		$\hat{\beta}$	0.381109	0.449112	0.411528		0.262668	0.304537	0.279459
		$\hat{\lambda}$	0.089828	0.113141	0.096816		0.0649630	0.0670547	0.0663601
	Bias	$\hat{\alpha}$	0.061023	0.043411	0.043550		0.0549728	0.041562	0.0474951
		$\hat{\beta}$	0.073021	0.005804	0.017642		0.039111	0.0017732	0.014487
		$\hat{\lambda}$	0.047069	0.031565	0.032979		0.0440804	0.0330376	0.0378170
N	**Est.**		MLE	LSE	WLSE	N	MLE	LSE	WLSE
150	Mean	$\hat{\alpha}$	2.456446	2.440770	2.445676	200	2.4565282	2.4406970	2.4456678
		$\hat{\beta}$	3.519232	3.489731	3.501626		3.514400	3.486704	3.4997339
		$\hat{\lambda}$	1.741641	1.729639	1.733615		1.7416867	1.7288745	1.7332505
	MSE	$\hat{\alpha}$	0.005195	0.004474	0.004404		0.0049965	0.0035919	0.0040484
		$\hat{\beta}$	0.046931	0.068878	0.056204		0.034643	0.048708	0.0397317
		$\hat{\lambda}$	0.003584	0.003052	0.003088		0.0031888	0.0023793	0.0027674
	RMSE	$\hat{\alpha}$	0.072077	0.066893	0.066366		0.0706865	0.0599332	0.0636272
		$\hat{\beta}$	0.216637	0.262446	0.237074		0.186127	0.220699	0.1993283
		$\hat{\lambda}$	0.059870	0.055246	0.055575		0.0564694	0.0487789	0.0526068
	Bias	$\hat{\alpha}$	0.056446	0.040770	0.045676		0.0565282	0.0406970	0.0456678
		$\hat{\beta}$	0.019232	0.010268	0.001626		0.014400	0.013295	0.0002660
		$\hat{\lambda}$	0.041641	0.029639	0.033615		0.0416867	0.028874	0.0332505

TABLE VI. MONTE CARLO SIMULATIONS FOR THE WT6E

$\alpha = 2.4$, $\beta = 3.5$, $\lambda = 1.7$									
N	**Est.**	**Ess. Par.**	**MLE**	**LSE**	**WLSE**	**N**	**MLE**	**LSE**	**WLSE**
50	Mean	$\hat{\alpha}$	2.454624	2.456688	2.451447	100	2.4586612	2.4435422	2.441039
		$\hat{\beta}$	3.630686	3.492248	3.527694		3.564998	3.519760	3.538074
		$\hat{\lambda}$	1.744410	1.739544	1.738138		1.7439088	1.7312234	1.7305331
	MSE	$\hat{\alpha}$	0.018311	0.023232	0.020556		0.0078340	0.0087219	0.011174
		$\hat{\beta}$	0.226517	0.281715	0.245636		0.097075	0.137016	0.115214
		$\hat{\lambda}$	0.011309	0.013348	0.012473		0.0053007	0.0058356	0.0073551
	RMSE	$\hat{\alpha}$	0.135319	0.152423	0.143376		0.0885104	0.0933915	0.105711
		$\hat{\beta}$	0.47593	0.530769	0.495617		0.311569	0.370157	0.339433
		$\hat{\lambda}$	0.106344	0.115534	0.111683		0.0728060	0.0763911	0.0857624
	Bias	$\hat{\alpha}$	0.054624	0.056688	0.051447		0.0586612	0.0435422	0.041039
		$\hat{\beta}$	0.130686	0.007751	0.027694		0.064998	0.019760	0.038074
		$\hat{\lambda}$	0.044410	0.039544	0.038138		0.0439088	0.0312234	0.0305331
N	**Est.**		MLE	LSE	WLSE	N	MLE	LSE	WLSE
150	Mean	$\hat{\alpha}$	2.454241	2.441826	2.443571	200	2.4554112	2.4416693	2.4467630
		$\hat{\beta}$	3.546014	3.507132	3.521989		3.518086	3.493963	3.5052263
		$\hat{\lambda}$	1.740693	1.730296	1.732213		1.7408544	1.730058	1.734258
	MSE	$\hat{\alpha}$	0.005726	0.006164	0.005835		0.0050189	0.0040316	0.0042625
		$\hat{\beta}$	0.061135	0.087888	0.072386		0.040129	0.059473	0.0472860
		$\hat{\lambda}$	0.003594	0.003829	0.003454		0.0030825	0.002362	0.002650
	RMSE	$\hat{\alpha}$	0.075671	0.078516	0.076389		0.0708443	0.0634953	0.0652880
		$\hat{\beta}$	0.247256	0.296459	0.269047		0.200323	0.243870	0.2174534
		$\hat{\lambda}$	0.059953	0.061881	0.058778		0.0555202	0.048609	0.051483
	Bias	$\hat{\alpha}$	0.054241	0.041826	0.043571		0.0554112	0.0416693	0.0467630
		$\hat{\beta}$	0.046014	0.007132	0.021989		0.018086	0.006036	0.0052263
		$\hat{\lambda}$	0.040693	0.030296	0.032213		0.0408544	0.030058	0.034258

TABLE VII. MONTE CARLO SIMULATIONS FOR THE WT7E

$\alpha = 2.4, \quad \beta = 3.5, \quad \lambda = 1.7$									
N	**Est.**	**Ess. Par.**	**MLE**	**LSE**	**WLSE**	**N**	**MLE**	**LSE**	**WLSE**
50	**Mean**	$\hat{\alpha}$	2.508692	2.489909	2.480260	100	2.515150	2.481503	2.49142
		$\hat{\beta}$	3.61137	3.492497	3.522203		3.561452	3.511073	3.528659
		$\hat{\lambda}$	1.693780	1.681322	1.688716		1.683553	1.6836987	1.6847023
	MSE	$\hat{\alpha}$	0.069011	0.150355	0.105481		0.037069	0.060109	0.192533
		$\hat{\beta}$	0.22484	0.284088	0.246636		0.093463	0.108228	0.108228
		$\hat{\lambda}$	0.010262	0.018574	0.014769		0.0074763	0.0063336	0.0063336
	RMSE	$\hat{\alpha}$	0.262700	0.387757	0.324779		0.192533	0.268595	0.245172
		$\hat{\beta}$	0.474175	0.532999	0.496624		0.305718	0.362882	0.328980
		$\hat{\lambda}$	0.101303	0.136286	0.121529		0.067329	0.0864657	0.0795841
	Bias	$\hat{\alpha}$	0.108692	0.089909	0.080260		0.115150	0.081503	0.091428
		$\hat{\beta}$	0.11137	0.007502	0.022203		0.061452	0.011073	0.0286596
		$\hat{\lambda}$	0.006219	0.018677	0.011283		0.016446	0.0163012	0.0152976
N	**Est.**		MLE	LSE	WLSE	N	MLE	LSE	WLSE
150	**Mean**	$\hat{\alpha}$	2.533397	2.503079	2.504083	200	2.521522	2.507171	2.495622
		$\hat{\beta}$	3.530690	3.499410	3.512302		3.531115	3.5009787	3.513974
		$\hat{\lambda}$	1.679426	1.680779	1.682712		1.6797045	1.6790027	1.683089
	MSE	$\hat{\alpha}$	0.034851	0.058090	0.044633		0.0257668	0.052068	0.039078
		$\hat{\beta}$	0.062387	0.089372	0.073177		0.0432339	0.0605698	0.049939
		$\hat{\lambda}$	.0029671	0.004417	0.003771		0.0021092	0.003396	0.0027723
	RMSE	$\hat{\alpha}$	0.186684	0.241020	0.211267		0.160520	0.228185	0.197683
		$\hat{\beta}$	0.249775	0.298952	0.270513		0.2079278	0.2461093	0.223470
		$\hat{\lambda}$	0.054471	0.066463	0.061414		0.0459270	0.0582765	0.0526529
	Bias	$\hat{\alpha}$	0.133397	0.103079	0.104083		0.1215228	0.107171	0.095622
		$\hat{\beta}$	0.030690	0.000589	0.012302		0.031115	0.0009787	0.0139747
		$\hat{\lambda}$	0.020573	0.019220	0.017287		0.0202954	0.0209972	0.0169100

TABLE VIII. MONTE CARLO SIMULATIONS FOR THE WT8E

$\alpha = 2.4, \quad \beta = 3.5, \quad \lambda = 1.7$									
N	**Est.**	**Ess. Par.**	**MLE**	**LSE**	**WLSE**	**N**	**MLE**	**LSE**	**WLSE**
50	**Mean**	$\hat{\alpha}$	2.503315	2.475502	2.477565	100	2.5148698	2.506437	2.492875
		$\hat{\beta}$	3.60398	3.517358	3.542295		3.559619	3.515364	3.532012
		$\hat{\lambda}$	1.692690	1.687538	1.690212		1.6849794	1.6814544	1.6859073
	MSE	$\hat{\alpha}$	0.049524	0.104801	0.081518		0.033642	0.087541	0.050178
		$\hat{\beta}$	0.155618	0.196372	0.169073		0.076943	0.119106	0.095685
		$\hat{\lambda}$	0.006708	0.01242	0.009600		0.003297	0.007777	0.0044941
	RMSE	$\hat{\alpha}$	0.222541	0.323731	0.285514		0.183419	0.295874	0.224006
		$\hat{\beta}$	0.39448	0.44313	0.411185		0.277387	0.345118	0.309331
		$\hat{\lambda}$	0.081902	0.111454	0.097983		0.0574212	0.0881881	0.0670383
	Bias	$\hat{\alpha}$	0.103315	0.075502	0.077565		0.114869	0.106437	0.0928754
		$\hat{\beta}$	0.103984	0.017358	0.042295		0.059619	0.015364	0.032012
		$\hat{\lambda}$	0.007309	0.012461	0.009787		0.0150205	0.0185455	0.0140926
N	**Est.**		MLE	LSE	WLSE	N	MLE	LSE	WLSE
150	**Mean**	$\hat{\alpha}$	2.523738	2.494273	2.491734	200	2.520862	2.489233	2.487219
		$\hat{\beta}$	3.533023	3.500844	3.514667		3.523894	3.495341	3.5066150
		$\hat{\lambda}$	1.679874	1.681940	1.684135		1.6780629	1.680127	1.6824232
	MSE	$\hat{\alpha}$	0.026782	0.055322	0.035907		0.024681	0.041544	0.033436
		$\hat{\beta}$	0.047115	0.070244	0.056088		0.036307	0.0515160	0.041475
		$\hat{\lambda}$	0.002309	0.004076	0.002758		0.0019043	0.0027459	0.0022589
	RMSE	$\hat{\alpha}$	0.163652	0.235208	0.189493		0.157103	0.2038249	0.182856
		$\hat{\beta}$	0.217061	0.265037	0.236830		0.190544	0.2269713	0.2036551
		$\hat{\lambda}$	0.048052	0.063847	0.052523		0.0436387	0.0524021	0.0475287
	Bias	$\hat{\alpha}$	0.123738	0.094273	0.091734		0.120862	0.089233	0.087219
		$\hat{\beta}$	0.033023	0.000844	0.014667		0.023894	0.0046584	0.0066150
		$\hat{\lambda}$	0.020125	0.018059	0.015864		0.0219370	0.0198729	0.0175767

TABLE IX. MONTE CARLO SIMULATIONS FOR THE WT9E

$\alpha = 2.4, \quad \beta = 3.5, \quad \lambda = 1.7$									
N	**Est.**	**Ess. Par.**	**MLE**	**LSE**	**WLSE**	**N**	**MLE**	**LSE**	**WLSE**
50	Mean	$\hat{\alpha}$	2.54654	2.430426	2.456984	100	2.465826	2.410965	2.428094
		$\hat{\beta}$	3.570760	3.61371	3.596977		3.581856	3.60127	3.583586
		$\hat{\lambda}$	1.718860	1.670530	1.683317		1.680969	1.662679	1.670948
	MSE	$\hat{\alpha}$	0.28925	0.356476	0.311426		0.103087	0.131695	0.110336
		$\hat{\beta}$	0.091178	0.21673	0.151585		0.032471	0.10425	0.057513
		$\hat{\lambda}$	0.065133	0.078923	0.070780		0.015339	0.029773	0.015996
	RMSE	$\hat{\alpha}$	0.53782	0.597056	0.558056		0.321072	0.362898	0.332169
		$\hat{\beta}$	0.301958	0.46554	0.389340		0.180199	0.32288	0.239820
		$\hat{\lambda}$	0.255212	0.280933	0.266046		0.123852	0.172550	0.126476
	Bias	$\hat{\alpha}$	0.14654	0.030426	0.056984		0.065826	0.010965	0.028094
		$\hat{\beta}$	0.070760	0.11371	0.096977		0.081856	0.10127	0.083586
		$\hat{\lambda}$	0.018860	0.029469	0.016682		0.019030	0.037320	0.029051
N	**Est.**		MLE	LSE	WLSE	N	MLE	LSE	WLSE
150	Mean	$\hat{\alpha}$	2.432108	2.401409	2.415905	200	2.425611	2.4021267	2.411788
		$\hat{\beta}$	3.587455	3.587473	3.579200		3.580285	3.564859	3.570938
		$\hat{\lambda}$	1.668765	1.662081	1.668523		1.6676108	1.669332	1.668753
	MSE	$\hat{\alpha}$	0.067838	0.094030	0.076228		0.056395	0.0790682	0.065613
		$\hat{\beta}$	0.025795	0.065706	0.041437		0.023903	0.043885	0.035337
		$\hat{\lambda}$	0.009634	0.016152	0.010661		0.0078478	0.014714	0.010303
	RMSE	$\hat{\alpha}$	0.260458	0.306644	0.276095		0.237478	0.2811907	0.256151
		$\hat{\beta}$	0.160610	0.256332	0.203561		0.154608	0.209488	0.187982
		$\hat{\lambda}$	0.098157	0.127092	0.103253		0.0885878	0.121304	0.101508
	Bias	$\hat{\alpha}$	0.032108	0.001409	0.015905		0.025611	0.0021267	0.011788
		$\hat{\beta}$	0.087455	0.087473	0.079200		0.080285	0.064859	0.070938
		$\hat{\lambda}$	0.031234	0.037918	0.031476		0.0323891	0.030667	0.031246

TABLE X. MONTE CARLO SIMULATIONS FOR THE WT10E

$\alpha = 2.4, \quad \beta = 3.5, \quad \lambda = 1.7$									
N	**Est.**	**Ess. Par.**	**MLE**	**LSE**	**WLSE**	**N**	**MLE**	**LSE**	**WLSE**
50	Mean	$\hat{\alpha}$	2.56169	2.43575	2.470037	100	2.4678520	2.421388	2.435439
		$\hat{\beta}$	3.566134	3.61186	3.600474		3.589869	3.596119	3.591859
		$\hat{\lambda}$	1.718334	1.663964	1.674867		1.677871	1.666898	1.669373
	MSE	$\hat{\alpha}$	0.26863	0.31937	0.276456		0.1172377	0.156708	0.130038
		$\hat{\beta}$	0.085390	0.199515	0.138450		0.04073	0.111363	0.065536
		$\hat{\lambda}$	0.058353	0.069144	0.051894		0.020295	0.034366	0.024108
	RMSE	$\hat{\alpha}$	0.518296	0.565129	0.525791		0.342400	0.395863	0.360608
		$\hat{\beta}$	0.292216	0.44667	0.372089		0.20181	0.333712	0.256000
		$\hat{\lambda}$	0.241563	0.262953	0.227804		0.142463	0.185382	0.155268
	Bias	$\hat{\alpha}$	0.161696	0.03575	0.070037		0.067852	0.021388	0.035439
		$\hat{\beta}$	0.066134	0.111863	0.10047		0.08986	0.096119	0.091859
		$\hat{\lambda}$	0.018334	0.036035	0.025132		0.022128	0.033101	0.030626
N	**Est.**		MLE	LSE	WLSE	N	MLE	LSE	WLSE
150	Mean	$\hat{\alpha}$	2.443062	2.413015	2.425092	200	2.417214	2.3908361	2.4008811
		$\hat{\beta}$	3.584717	3.589792	3.576343		3.582335	3.585214	3.577797
		$\hat{\lambda}$	1.671255	1.662509	1.669996		1.668373	1.660333	1.6660111
	MSE	$\hat{\alpha}$	0.072969	0.095717	0.079686		0.047897	0.0685325	0.0548017
		$\hat{\beta}$	0.029814	0.077951	0.042268		0.024880	0.050875	0.036844
		$\hat{\lambda}$	0.012857	0.017067	0.011918		0.009251	0.011211	0.0079173
	RMSE	$\hat{\alpha}$	0.270128	0.309383	0.282287		0.218855	0.2617871	0.2340978
		$\hat{\beta}$	0.172668	0.279198	0.205593		0.157735	0.225556	0.191948
		$\hat{\lambda}$	0.113390	0.130642	0.109172		0.096183	0.105885	0.0889795
	Bias	$\hat{\alpha}$	0.043062	0.013015	0.025092		0.017214	0.0091638	0.0008811
		$\hat{\beta}$	0.084717	0.089792	0.076343		0.082335	0.085214	0.077797
		$\hat{\lambda}$	0.028744	0.037490	0.030003		0.031626	0.039666	0.0339888

TABLE XI. MONTE CARLO SIMULATIONS FOR THE WT11E

				$\alpha = 2.4$, $\beta = 3.5$, $\lambda = 1.7$					
N	**Est.**	**Ess. Par.**	**MLE**	**LSE**	**WLSE**	**N**	**MLE**	**LSE**	**WLSE**
50	Mean	$\hat{\alpha}$	2.364539	2.735233	2.598566	100	2.408658	2.59669	2.524323
		$\hat{\beta}$	3.649153	3.430030	3.497050		3.66746	3.538035	3.580815
		$\hat{\lambda}$	1.747772	1.666009	1.694930		1.692662	1.6533659	1.6708908
	MSE	$\hat{\alpha}$	0.565000	1.64324	1.045475		0.3028497	0.688192	0.5381097
		$\hat{\beta}$	0.536360	1.091212	0.937202		0.284689	0.4735789	0.3538143
		$\hat{\lambda}$	0.052416	0.10719	0.085919		0.0219164	0.0564526	0.0412849
	RMSE	$\hat{\alpha}$	0.751665	1.28189	1.02248		0.550317	0.82957	0.733559
		$\hat{\beta}$	0.732366	1.044611	0.968092		0.533562	0.688170	0.594822
		$\hat{\lambda}$	0.228945	0.32740	0.293120		0.148042	0.237597	0.203186
	Bias	$\hat{\alpha}$	0.035460	0.335233	0.198566		0.008658	0.196691	0.124323
		$\hat{\beta}$	0.149153	0.069969	0.002949		0.16746	0.038035	0.080815
		$\hat{\lambda}$	0.047772	0.03399	0.005069		0.007337	0.046634	0.029109
N	**Est.**		MLE	LSE	WLSE	N	MLE	LSE	WLSE
150	Mean	$\hat{\alpha}$	2.442790	2.567411	2.515603	200	2.441836	2.52982	2.502283
		$\hat{\beta}$	3.645084	3.551368	3.584229		3.65198	3.559541	3.60918
		$\hat{\lambda}$	1.673407	1.648710	1.661589		1.669717	1.653683	1.656051
	MSE	$\hat{\alpha}$	0.207443	0.459532	0.336705		0.174382	0.309730	0.25928
		$\hat{\beta}$	0.202795	0.323247	0.242720		0.16738	0.269025	0.19271
		$\hat{\lambda}$	0.016210	0.034194	0.030018		0.012600	0.024341	0.020411
	RMSE	$\hat{\alpha}$	0.455459	0.67788	0.58026		0.417591	0.55653	0.509201
		$\hat{\beta}$	0.45032	0.568548	0.492666		0.409122	0.518676	0.438988
		$\hat{\lambda}$	0.127321	0.184918	0.173259		0.112252	0.156018	0.142867
	Bias	$\hat{\alpha}$	0.042790	0.167411	0.115603		0.041836	0.12982	0.102283
		$\hat{\beta}$	0.14508	0.051368	0.084229		0.151985	0.059541	0.109188
		$\hat{\lambda}$	0.026592	0.051289	0.038410		0.030283	0.046316	0.043948

TABLE XII. MONTE CARLO SIMULATIONS FOR THE WT12E

				$\alpha = 2.4$, $\beta = 3.5$, $\lambda = 1.7$					
N	**Est.**	**Ess. Par.**	**MLE**	**LSE**	**WLSE**	**N**	**MLE**	**LSE**	**WLSE**
50	Mean	$\hat{\alpha}$	2.468524	2.409045	2.424433	100	2.448050	2.4070865	2.422152
		$\hat{\beta}$	3.590128	3.62374	3.60595		3.590072	3.586244	3.584410
		$\hat{\lambda}$	1.678225	1.650366	1.660701		1.666596	1.657445	1.663770
	MSE	$\hat{\alpha}$	0.136800	0.190506	0.155590		0.083765	0.1167711	0.094510
		$\hat{\beta}$	0.058579	0.175432	0.10913		0.034075	0.071091	0.063649
		$\hat{\lambda}$	0.034395	0.040849	0.034262		0.013857	0.019240	0.017949
	RMSE	$\hat{\alpha}$	0.369865	0.436470	0.394449		0.289422	0.3417179	0.307424
		$\hat{\beta}$	0.242031	0.41884	0.33034		0.184594	0.266630	0.252288
		$\hat{\lambda}$	0.185460	0.202112	0.185101		0.117718	0.138708	0.133974
	Bias	$\hat{\alpha}$	0.068524	0.009045	0.024433		0.048050	0.0070865	0.022152
		$\hat{\beta}$	0.090128	0.12374	0.10595		0.090072	0.086244	0.084410
		$\hat{\lambda}$	0.021774	0.049633	0.039298		0.033403	0.042554	0.036229
N	**Est.**		MLE	LSE	WLSE	N	MLE	LSE	WLSE
150	Mean	$\hat{\alpha}$	2.440417	2.413413	2.423993	200	2.439461	2.417458	2.429093
		$\hat{\beta}$	3.580046	3.579031	3.570454		3.574830	3.568870	3.555546
		$\hat{\lambda}$	1.673421	1.668179	1.673515		1.6735221	1.670772	1.6796654
	MSE	$\hat{\alpha}$	0.057218	0.082153	0.066426		0.046165	0.063081	0.053728
		$\hat{\beta}$	0.023212	0.060548	0.036166		0.018797	0.039146	0.027414
		$\hat{\lambda}$	0.009826	0.014029	0.008865		0.0064154	0.012428	0.0069665
	RMSE	$\hat{\alpha}$	0.239203	0.286624	0.257733		0.214860	0.251161	0.231793
		$\hat{\beta}$	0.152357	0.246065	0.190174		0.137105	0.197855	0.165573
		$\hat{\lambda}$	0.099130	0.118445	0.094158		0.0800967	0.111483	0.0834660
	Bias	$\hat{\alpha}$	0.040417	0.013413	0.023993		0.039461	0.017458	0.029093
		$\hat{\beta}$	0.080046	0.079031	0.070454		.074830	0.068870	0.055546
		$\hat{\lambda}$	0.026578	0.031820	0.026484		0.0264778	0.029227	0.0203345

4. APPLICATION

To determine the efficiency of the previously presented distributions, a practical applications is conducted on real data represented by recovery times (in months) for a random sample of 128 bladder cancer patients reported in Lee and Wang (2003) [21], where comparison is conducted using four information criteria which are (AIC [34] [35], CAIC [29] [36], BIC [37] [38], and HQIC [39] [40]) in addition to four statistical measures, which are: Kolmogorov-Smirnov (KS) [41] [42], Anderson- Darling (A) [43], Cramér-von Mises (W) [44] [45], and p-value [23], [2].

Table 13 shows the value of the informatics criteria for different distributions, while table 14 shows the values of the statistical measures for the different distributions, while table 15 shows the parameters estimator by MLE for the different distributions.

TABLE XII. INFORMATICS CRITERIA FOR DIFFERENT DISTRIBUTIONS

Dist.	-Log (L)	AIC	CAIC	BIC	HQIC
WT1E	410.5979	827.1958	827.3893	835.7518	830.6721
WT2E	408.9357	823.8715	823.9032	826.7235	825.0303
WT3E	408.0775	822.155	822.3485	830.7111	825.6314
WT4E	410.5979	827.1958	827.3893	835.7519	830.6722
WT5E	412.2369	830.4738	830.6674	839.0299	833.9502
WT6E	409.5265	825.0581	825.2516	833.6141	828.5344
WT7E	413.4038	832.8104	833.004	841.3665	836.2868
WT8E	408.6789	823.3637	823.5573	831.9198	826.8401
WT9E	413.6026	833.2084	833.402	841.7645	836.6848
WT10E	435.2738	876.5476	876.7411	885.1037	880.024
WT11E	413.7953	833.636	833.8295	842.1921	837.1124
WT12E	406.0376	818.0751	818.2687	826.6312	821.5515

TABLE XIV. STATISTICAL MEASURES FOR THE DIFFERENT DISTRIBUTIONS

Dist.	W	A	K-S	p-value
WT1E	0.1410266	0.8607159	0.07160868	0.5277046
WT2E	0.1410655	0.8609474	0.0715582	0.5286236
WT3E	0.1270024	0.7770797	0.9995304	0
WT4E	0.141083	0.861052	0.07166681	0.5266472
WT5E	0.170906	1.041197	0.07782965	0.4201514
WT6E	0.1204754	0.7379188	0.06915452	0.5730348
WT7E	0.2017909	1.246293	0.08132049	0.3656726
WT8E	0.09979402	0.6123915	0.06767212	0.6009415
WT9E	38.86583	252.302	0.9999012	0
WT10E	0.6057257	3.577908	0.2063557	3.687338e-05
WT11E	38.74645	252.1637	0.9999402	0
WT12E	0.03192105	0.2037036	0.03723675	0.9943027

TABLE XV. PARAMETERS ESTIMATOR BY MLE FOR THE DIFFERENT DISTRIBUTIONS

Dist.	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$
WT1E	1.0773631	1.0552710	0.1153875
WT2E	4.01100346	1.05537765	0.02872513
WT3E	0.58869240	1.50503242	0.06323829
WT4E	1.05549023	0.11395114	0.01220293
WT5E	15.779281210	0.994822351	0.006348757
WT6E	0.0170908	1.1744118	0.1217004
WT7E	1.5882521	0.5045123	0.2329852
WT8E	1.0462806	1.1757085	0.1055157
WT9E	1.5226857	0.3598340	0.3572202
WT10E	0.6190966	0.1915228	0.5235674
WT11E	1.3366383	0.2149499	0.1096531
WT12E	1.1796094	0.7292086	0.1365605

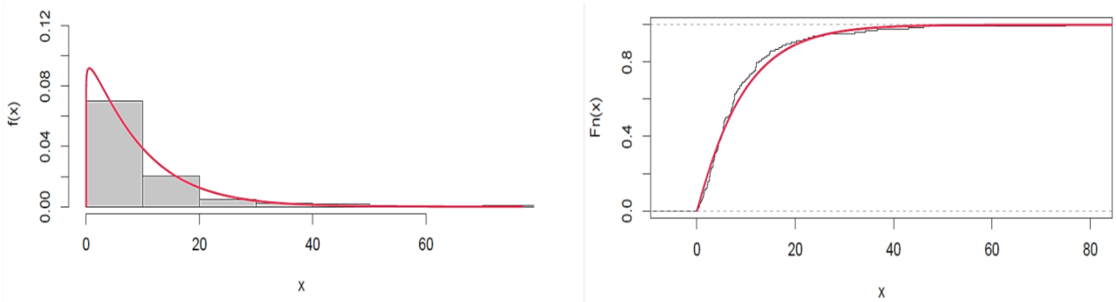


Fig. 15. Fitted density and CDF for WT1E with Data set

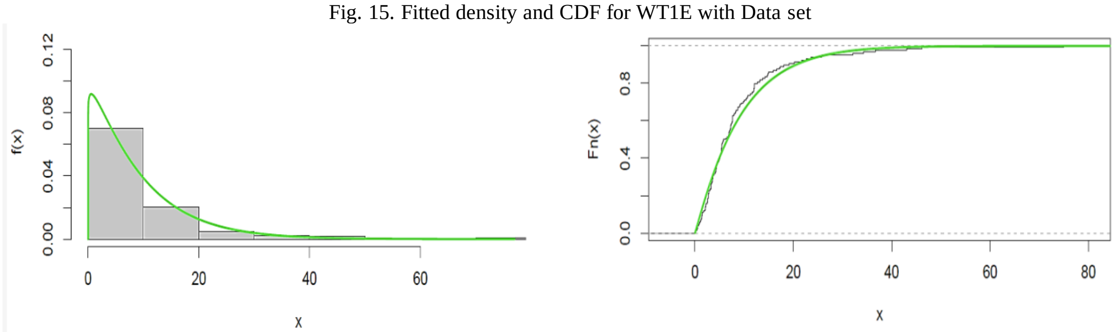


Fig. 16. Fitted density and CDF for WT2E with Data set

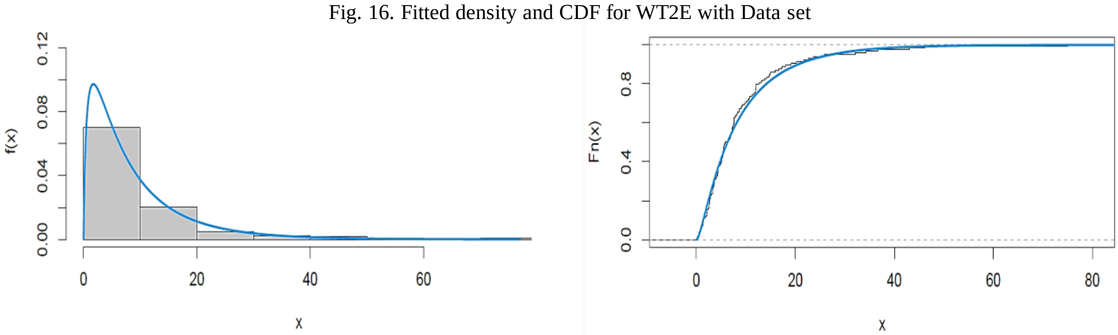


Fig. 17. Fitted density and CDF for WT3E with Data set

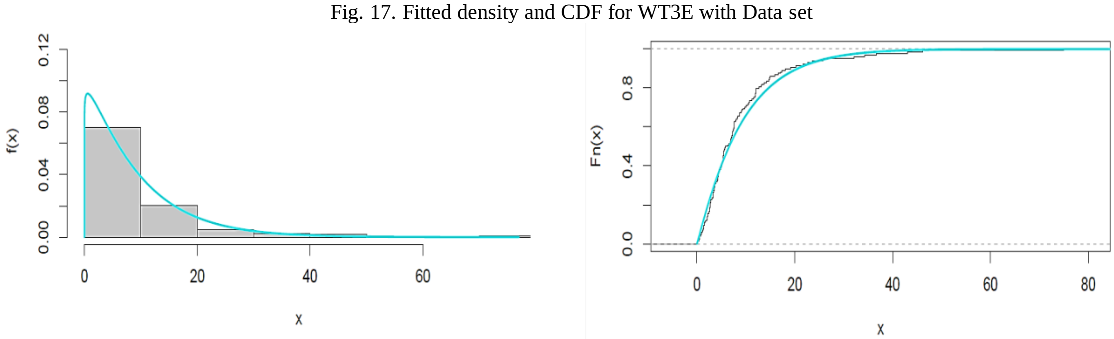


Fig. 18. Fitted density and CDF for WT4E with Data set

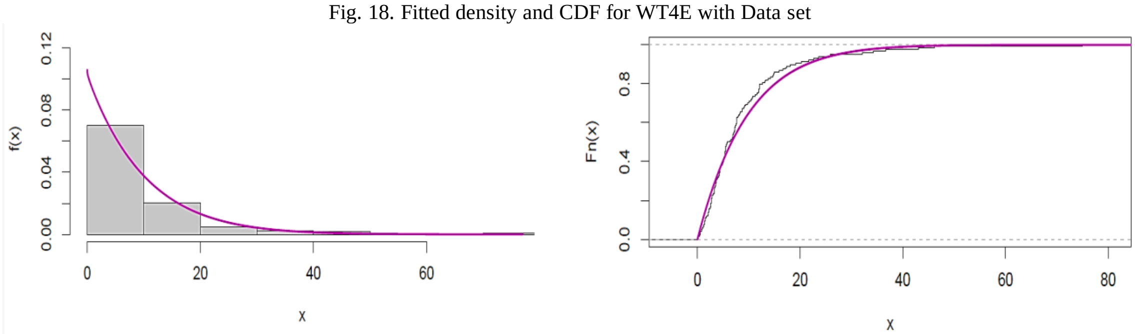


Fig. 19. Fitted density and CDF for WT5E with Data set

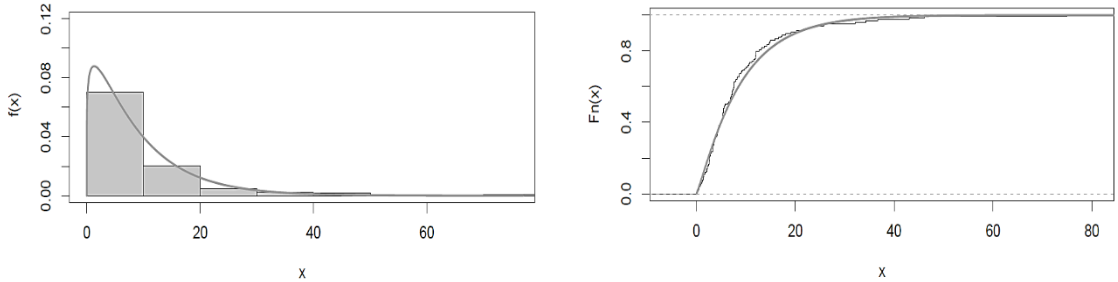


Fig. 20. Fitted density and CDF for WT6E with Data set

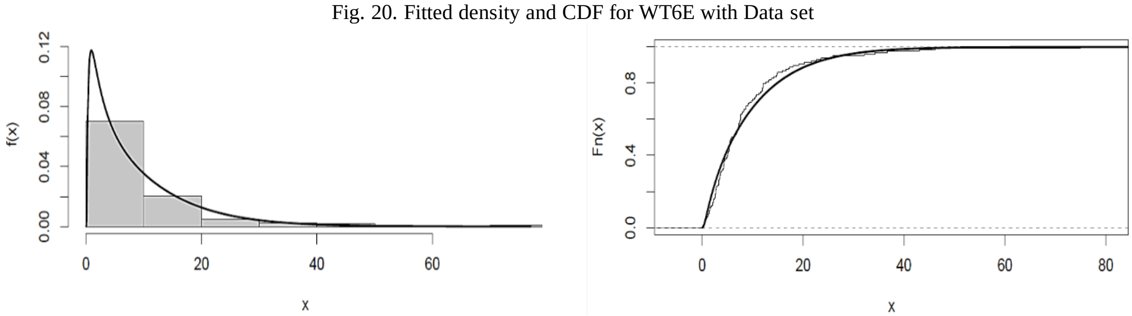


Fig. 21. Fitted density and CDF for WT7E with Data set

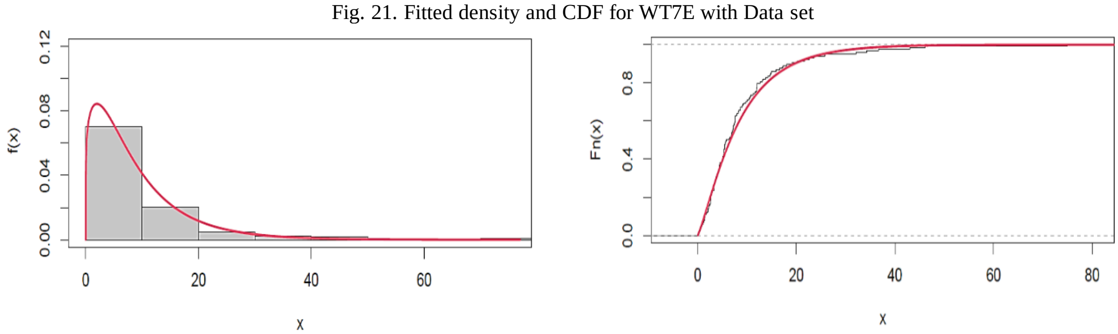


Fig. 22. Fitted density and CDF for WT8E with Data set

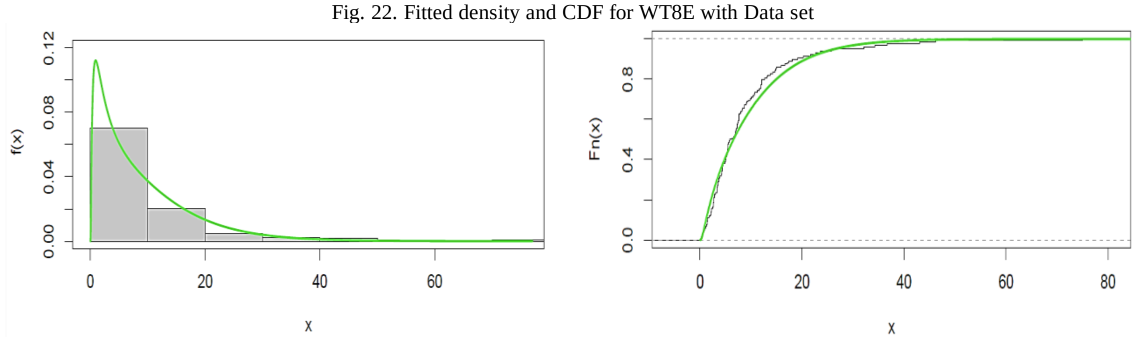


Fig. 23. Fitted density and CDF for WT9E with Data set

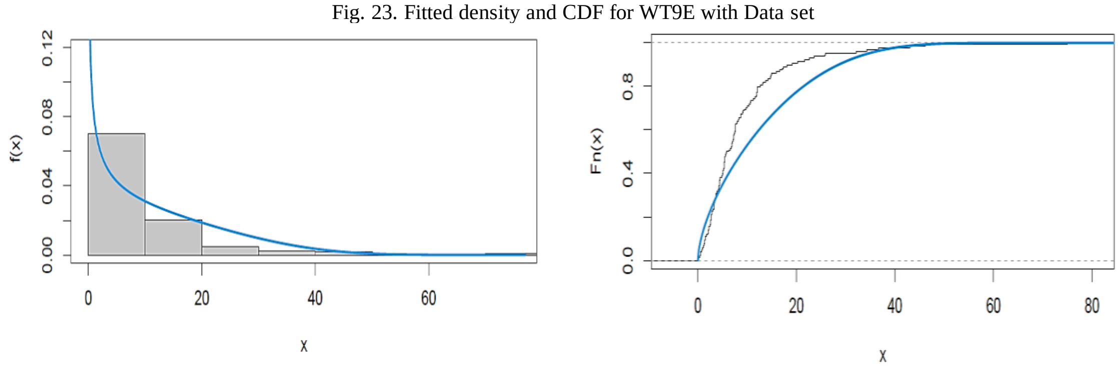


Fig. 24. Fitted density and CDF for WT10E with Data set

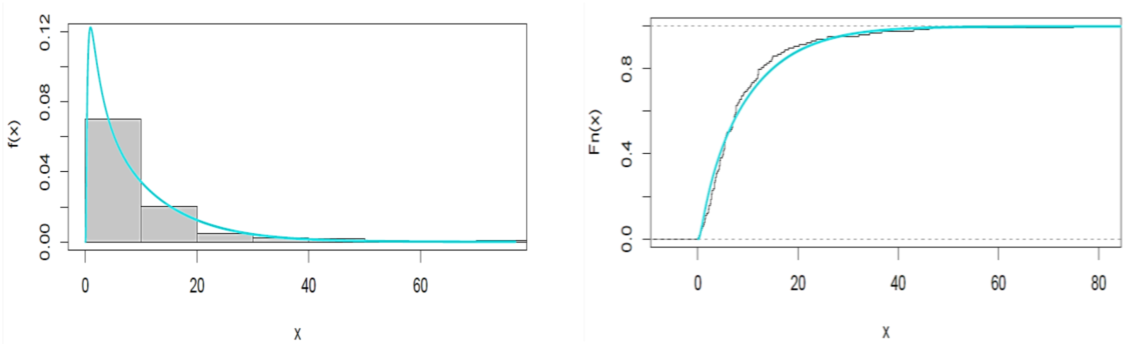


Fig. 25. Fitted density and CDF for WT11E with Data set

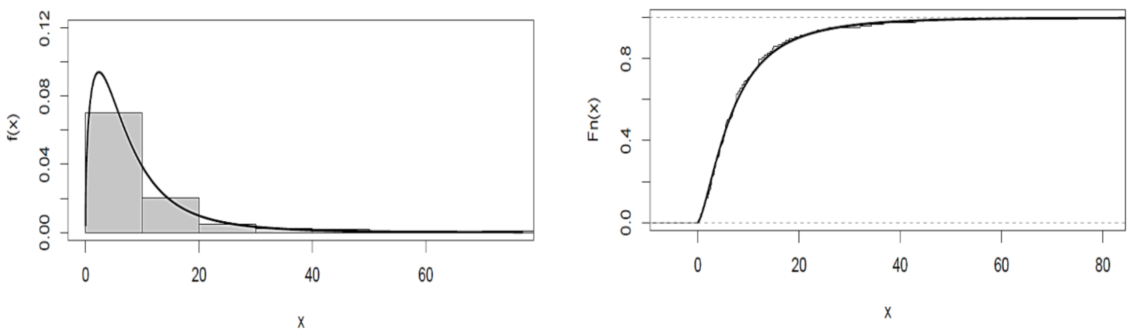


Fig. 26. Fitted density and CDF for WT12E with Data set

5. RESULTS AND DISCUSSION

The results obtained in this study were divided into three parts: structural of family, simulation, and practical application, as shown below:

1. WT1
    - **structural of family**: relatively simple with basic equations. Easley implementable using conventional estimation techniques.
    - **Simulation accuracy**: shows moderate MSE and RMSE values, indicating average simulation performance.
    - **Practical application**: shows good results in K-S and A tests with moderate results in information criteria (AIC, BIC).
2. WT2
    - **structural of family**: similar to WT1 but has some modifications that add additional flexibility.
    - **Simulation accuracy**: similar performance to WT1 with slight improvement in error reduction.
    - **Practical application**: achieved slightly better rustles in information criteria, making it relatively more efficient in some applications.
3. WT3
    - **structural of family**: complex due to reliance on multiple integrals and increased number of parameters.
    - **Simulation accuracy**: shows higher values of variance and albedo , which reduces its accuracy with small samples.
    - **Practical application**: it didn't perform well in the fit tests (K-S, and A) and the AIC and BIC values were high.
4. WT4
    - **structural of family**: it relies on minor improvements compared to WT1, which improves its adaptive ability.
    - **Simulation accuracy**: it showed acceptable results in error values, bit it was not the best.
    - **Practical application**: results similar to WT1 with a slight advantage in the fit tests.

5. WT5
    - **structural of family**: complex due to addition of new parameters that increase flexibility.
    - **Simulation accuracy**: it showed good results with large samples but was less stable with small samples.
    - **Practical application**: it showed modest performance in the fit tests with relatively good results in the information criteria.
6. WT6
    - **structural of family**: balanced between simplicity and flexibility.
    - **Simulation accuracy**: it showed the lowest error values (MSE, RMSE), reflecting high simulation accuracy.
    - **Practical application**: it achieved excellent performance in all fit tests and information criteria, making it one of the most efficient.
7. WT7
    - **structural of family**: complex due to reliance on inverse functions.
    - **Simulation accuracy**: it showed relatively high error values with high variance.
    - **Practical application**: it performed poorly in fit tests and information criteria, which reduces its effectiveness.
8. WT8
    - **structural of family**: relatively simple with sufficient flexibility.
    - **Simulation accuracy**: it showed stable results and low error values. .
    - **Practical application**: it performed very well in the fit tests (K-S, and A), with good results in information criteria.
9. WT9
    - **structural of family**: relatively complex with the addition of new parameters.
    - **Simulation accuracy**: it showed high error values, especially with small samples.
    - **Practical application**: it didn't achieve good results in fit tests and the AIC and BIC values were high.
10. WT10
    - **structural of family**: most complex due to the use the inverse and modified functions.
    - **Simulation accuracy**: it showed the highest error values, making it the least efficient in simulation.
    - **Practical application**: very poor performance in fit tests and information criteria.
11. WT11
    - **structural of family**: very complex due to reliance on new mathematical additions.
    - **Simulation accuracy**: it showed modest results with higher values of variance.
    - **Practical application**: relatively poor performance in fit tests and the AIC and BIC values were high.
12. WT12
    - **structural of family**: balanced between complexity and flexibility.
    - **Simulation accuracy**: it showed the lowest error values and was the best in terms of stability.
    - **Practical application**: it achieved the best performance in all tests and criteria, making it the most efficient.

6. COMPARISION

1. Structural comparison:
    - Structural complexity: the twelve families vary in the complexity of their equations; families such as WT3 and WT5 have equations with high mathematical complexity due to the introduction of additional factors such as integral or inverse Structures. While families such as WT1 and WT2 are relatively simple in their Structural Structure.

- Computational difficulties: the difficulty of estimation increases with the increase in the number of parameters and integral constraints. The WT11 family is considered the most complex, requiring greater computational resources.

2. Simulation accuracy:
   - Error measures: the tables indicated that families WT6 and WT12 showed the lowest values for error measures such as MSE and RMSE, indicating the ability of these families to simulate data more accurately.
   - Variance low albedo: the WT3 family suffered from high variance and relatively low albedo, making it not ideal for use with small samples.
   - Estimation Efficiency: according to the three estimation methods, families such as WT8 showed relative stability with all sample size.
3. Practical application efficiency:
   - Fitness tests: the WT12 family achieved the best results in the K-S, A, and w tests, making it the most suitable for real data applications.
   - Information criteria: the WT12 family obtained the lowest values in AIC, BIC, and HQIC, indicating its ability to provide more efficient and simple models without sacrificing accuracy.
   - Practical applications: data extracted from bladder cancer patients showed a high agreement with WT12 and WT6, reflecting the efficiency of these families in real applications compared to others.

At the best end in terms of simulation are WT6, WT12, while the weakest are WT10, WT7. While the efficiency of practical application the best are WT12 and WT6, while the weakest is WT10 and WT11.

7. CONCLUSIONS

The most efficient families are WT12 and WT6 have the best balance between structural simplicity, simulation accuracy, and practical application efficiency. The medium families are WT1, WT2, and WT8 are suitable for applications that require simplicity with reasonable accuracy. The weaker families are WT10 and WT11 suffer from increased complexity without offering significant applications that require high performance with real data, and the WT6 is used as a balance between flexibility and accuracy.

Conflicts of Interest

The authors declare no conflicts of interest

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Non

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